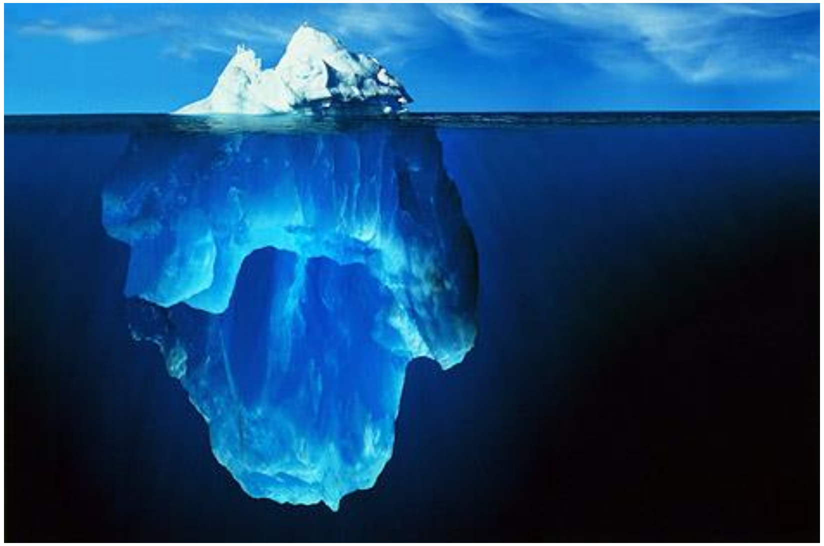
MIP* = RE
and
Tsirelson's problem



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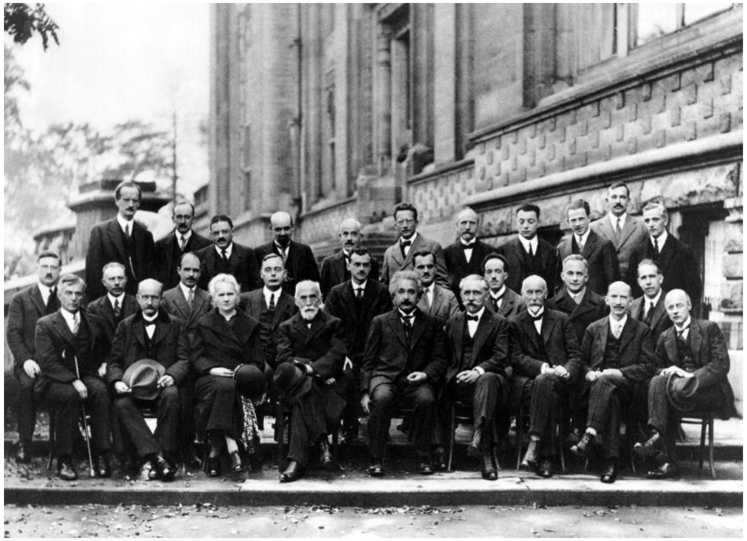
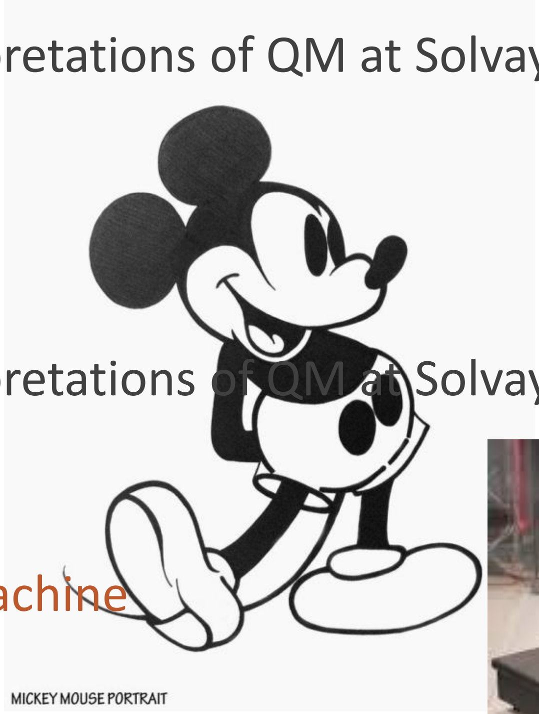
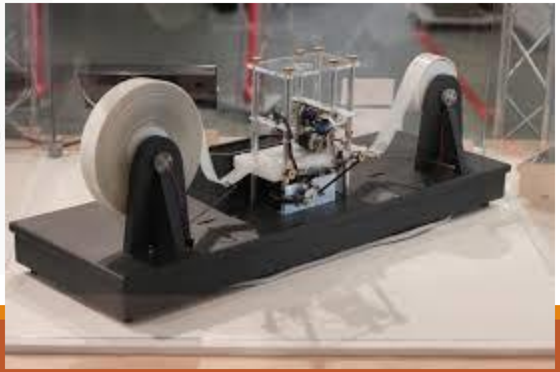
Joint work with Zhengfeng Ji (UTS Sydney), Anand Natarajan (Caltech), John Wright (Caltech/UT Austin) and Henry Yuen (Toronto)

A hundred years ago…



- 1922: Louis Armstrong starts improvising
- 1925: Heisenberg develops matrix mechanics
- 1927: Lindbergh flight from New York to Paris
- 1927: Einstein, Bohr debate interpretations of QM at Solvay conference *in person*
- 1928: Mickey Mouse
- 1929: Stock market crashes
- 1930: Einstein, Bohr debate interpretations of QM at Solvay conference *in person*
- 1935: EPR paradox paper
- 1936: Turing invents the Turing machine

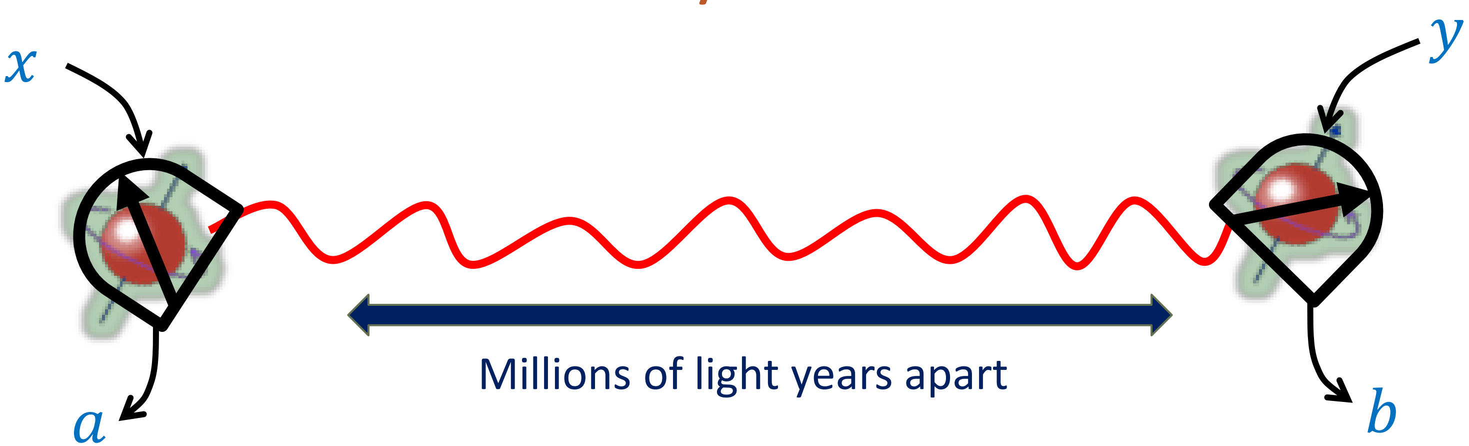




Fast forward 100 years

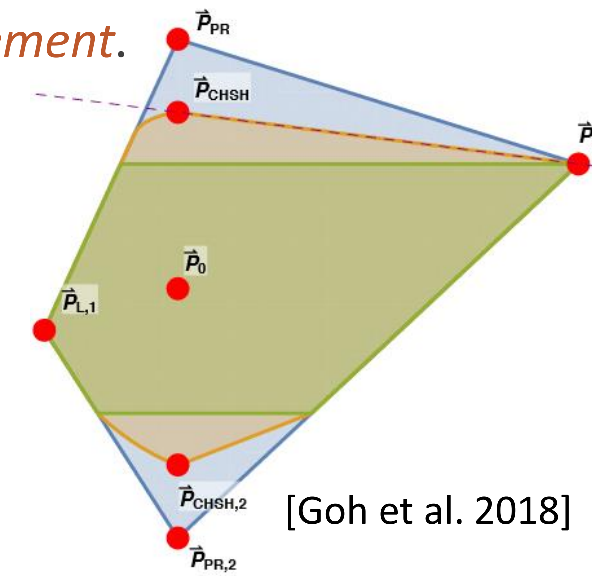
- Matrix mechanics has developed into the theory of operator algebras (von Neumann, Connes, Kirchberg, ….)
- Debates on the foundations of quantum mechanics are set aside in favor of impressive successes of quantum information & computation (quantum algorithms, quantum cryptography, new materials, ….)
- Computing devices are ubiquitous;
  Computer science becomes leading major at colleges worldwide
- All social interactions are Zoomified
- This talk: How operator algebras, quantum information, and the theory of computation reunite around a 3 decades-old question

Quantum nonlocality



Local measurements on far-away particles can exhibit unexpected correlations

Schrödinger called this phenomenon *quantum entanglement*.



[Goh et al. 2018]

Bell'1964: quantum mechanics is *nonlocal*

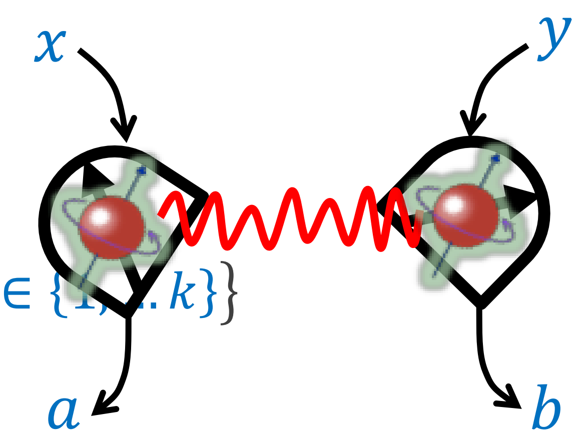
- Some correlations $\{p(a,b|x,y)\}$ have a model in QM but no classical (LHV) explanation
- [CHSH'69,…]: Bell inequalities separate quantum and classical correlations

Nonlocal correlations

The existence of nonlocal correlations has been experimentally verified multiple times over the years.

TU Delft, Netherlands (2015)

Tsirelson's setup



- *Correlation*: family of distributions

$$\{\, p(a,b|x,y) \mid\ x,y \in \{1,\dots,n\}\ \ a,b \in \{1,\dots k\}\}$$

- When is a correlation classical? When is it quantum?

- Classical correlations:

$$p(a,b|x,y) = \int_\lambda q_A(a|x,\lambda)q_B(b|y,\lambda)d\lambda$$

- Quantum correlations?

Tsirelson in the '80s introduces two possible representations:

$$p(a,b|x,y) = \langle\psi|P_a^x \otimes Q_b^y|\psi\rangle : \quad |\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$$

$$p(a,b|x,y) = \langle\psi|P_a^x Q_b^y|\psi\rangle: \quad |\psi\rangle \in \mathcal{H}, [P_a^x, Q_b^y] = 0$$

Tsirelson’s problem

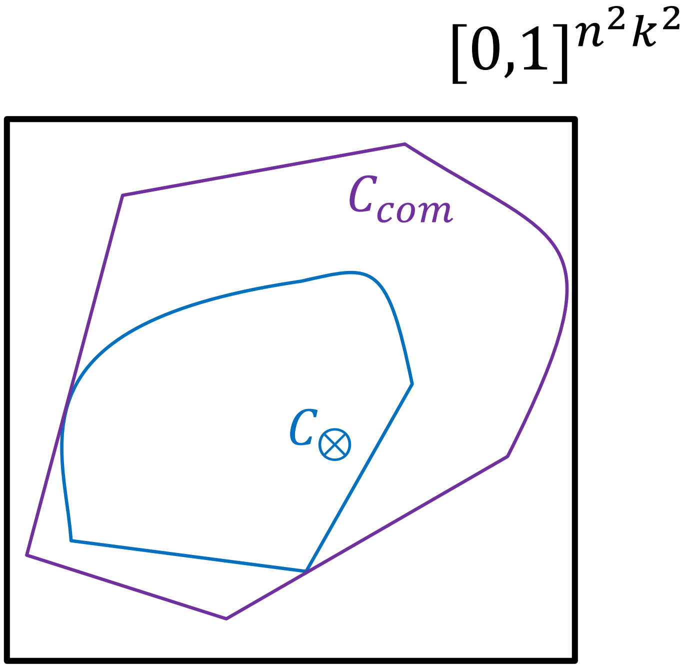
Quantum Bell-type inequalities are defined in terms of two (or more) subsystems of a quantum system. The subsystems may be treated either via (local) Hilbert spaces, - tensor factors of the given (global) Hilbert space, or via commuting (local) operator algebras. The latter approach is less restrictive, it just requires that the given operators commute whenever they belong to different subsystems.

Are these two approaches equivalent?

Tsirelson's problem

$C_{\otimes}(n,k) = \{ \left(\langle\psi|P_a^x \otimes Q_b^y|\psi\rangle\right)_{abxy} : \ |\psi\rangle \in \mathcal{H} \otimes \mathcal{H} \}$

$C_{com}(n,k) = \{ \left(\langle\psi|P_a^x Q_b^y|\psi\rangle\right)_{abxy} :$
$|\psi\rangle \in \mathcal{H}, [P_a^x, Q_b^y] = 0\}$



- Both sets are convex
- $C_{\otimes}(n,k) \subseteq C_{com}(n,k)$ for all n, k.
- Equality holds if dim($\mathcal{H}$)$< \infty$
- $C_{com}(n,k)$ is closed, but [Slofstra'18] $C_{\otimes}(n,k)$ is not!

Is $\overline{C_{\otimes}(n,k)} = C_{com}(n,k)$ for all $n, k \geq 2$?

Why do we care?

Is $\overline{C_{\otimes}(n,k)} = C_{com}(n,k)$ for all $n, k \geq 2$?

- Don't we need to know how to model locality in quantum mechanics?
- Connes' 1976 "Embedding Problem" (CEP) :

  "Every type II_1 von Neumann algebra embeds in an ultrapower of the hyperfinite II_1 factor $\mathcal{R}$"

- Kirchberg's 1993 QWEP conjecture:

  $$C^*(F_2) \otimes_{min} C^*(F_2) = C^*(F_2) \otimes_{max} C^*(F_2)$$

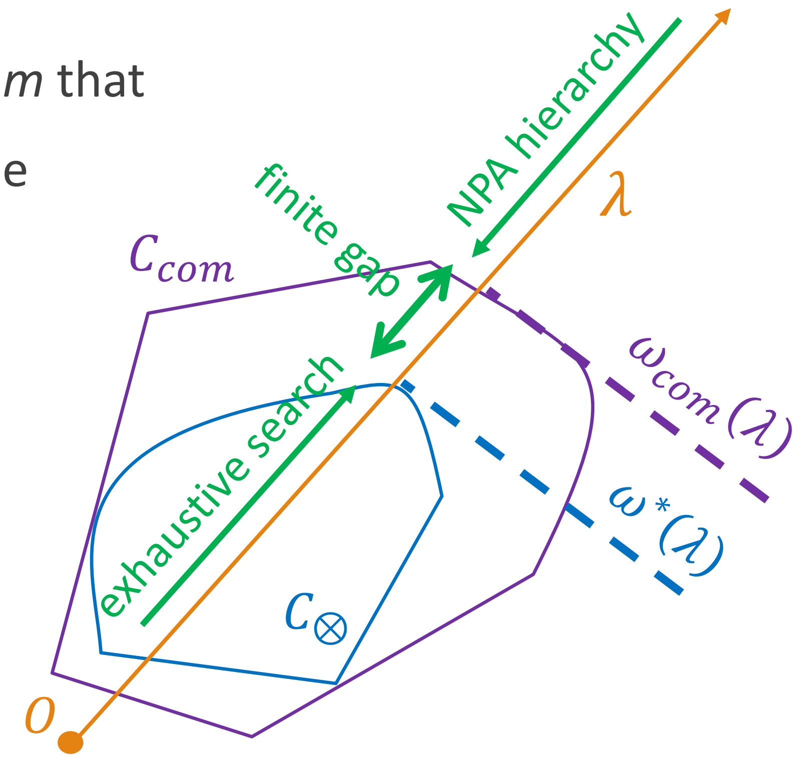
- [Kirchberg, Fritz, Junge et al., Ozawa]: CEP $\leftrightarrow$ QWEP $\leftrightarrow$ Tsirelson's problem
- Multiple reformulations: free entropy, group theory, etc.

A negative resolution

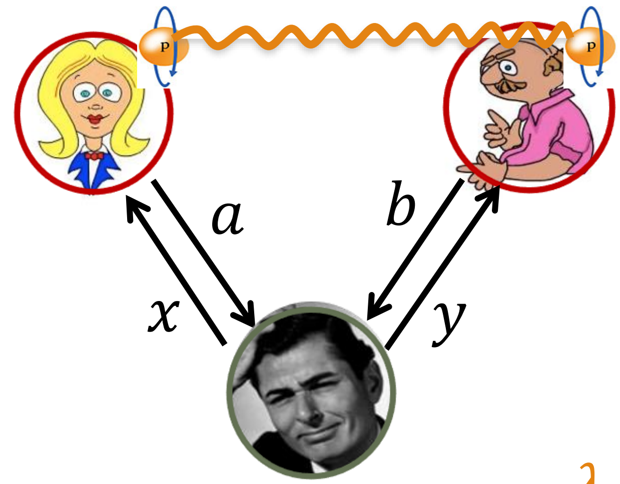
- [JNVWY'20] There are finite n, k such that $\overline{C_{\otimes}(n,k)} \subsetneq C_{com}(n,k)$

- We show a stronger result: there is *no algorithm* that given coefficients λ for an n-setting, k-outcome Bell inequality computes (even approximately) the *quantum value*

$$\omega^*(\lambda) = \sup_{p \in C_{\otimes}(n,k)} |\lambda \cdot p|$$
$$= \sup_{p \in C_{\otimes}(n,k)} \left| \sum_{xyab} \lambda_{xyab} \, p(a,b|x,y) \right|$$



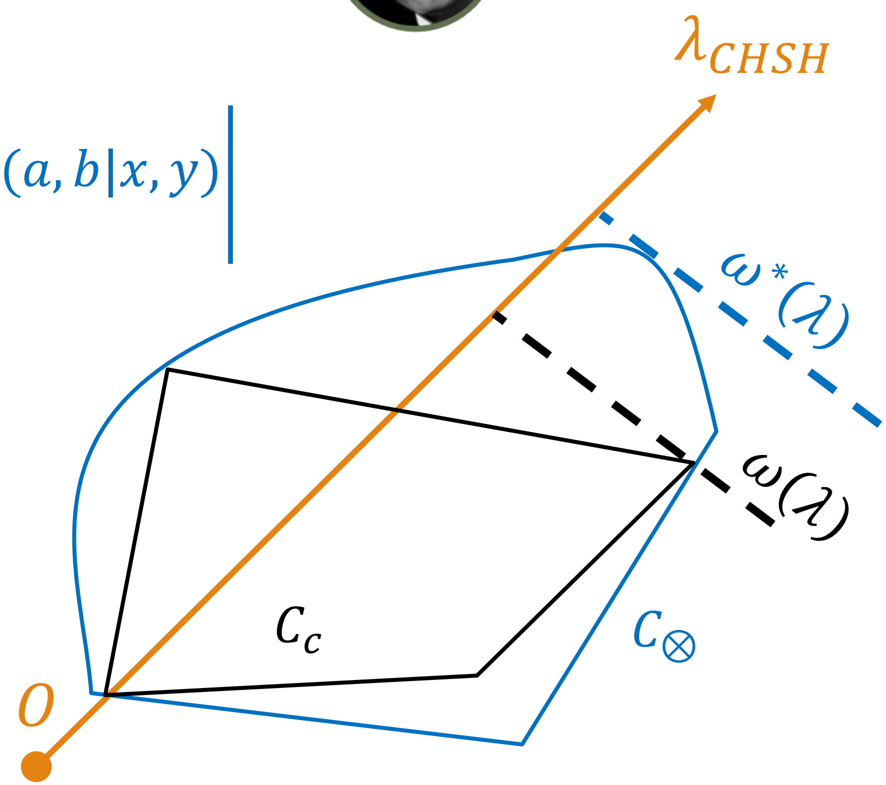
Nonlocal games



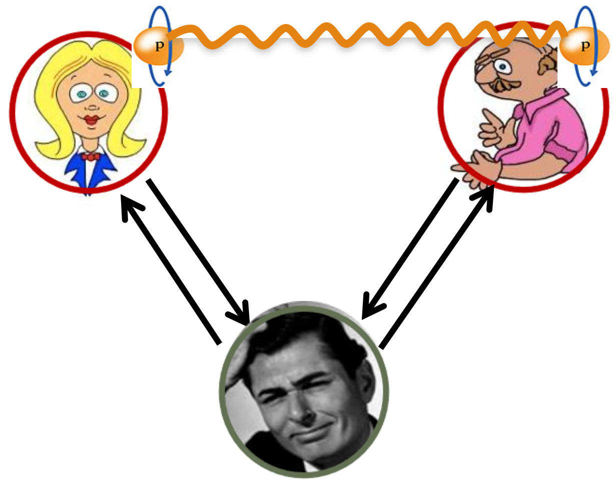
- A n-input, k-output nonlocal game G is:
  - A distribution π on $\{1, \dots, n\} \times \{1, \dots, n\}$
  - A game predicate $V(a, b|x, y) \in \{0,1\}$
- $(\lambda_G)_{abxy} = \pi(x, y)V(a, b|x, y)$

$$\omega^*(G) = \sup_{p \in C_\otimes(n,k)} \left| \sum_{xyab} \pi(x, y)V(a, b|x, y)\, p(a, b|x, y) \right|$$

- Ex: CHSH game
  - $n = k = 2$
  - $\pi(x, y) = 1/4$ for all $x, y \in \{0,1\}$
  - $V(a, b|x, y) = 1$ iff $a \oplus b = x \wedge y$



From computation to games



- Map Turing Machine $M \rightarrow$ game G_M s.t:
  - If M halts then $\omega^*(G_M) = 1$
  - If M does not halt then $\omega^*(G_M) < \frac{1}{2}$

1. Embedding computational problems in games:
   Complexity class MIP* = quantum multiprover interactive proof systems
   [IV'12] NEXP ⊆ MIP*
   [NW'18] NEEXP ⊆ MIP*
   Use techniques from complexity theory: error-correcting codes, proof checking, etc.

2. Designing games that force high entanglement: self-testing
   [SW'88,MYS'12] Robust analysis of CHSH inequality
   [NV'17] Quantum linearity test

3. New ingredient: recursive compression of games

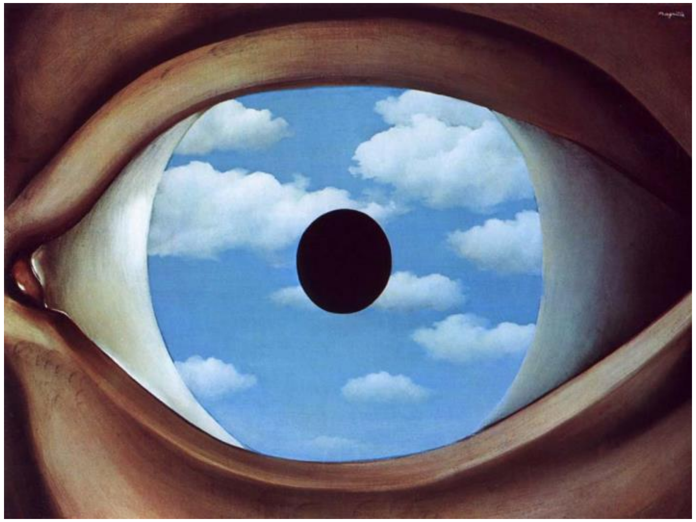
Summary

$C_\otimes(n,k) = \{ \left(\langle\psi|P_a^x \otimes Q_b^y|\psi\rangle\right)_{abxy} :\ |\psi\rangle \in \mathcal{H} \otimes \mathcal{H} \}$

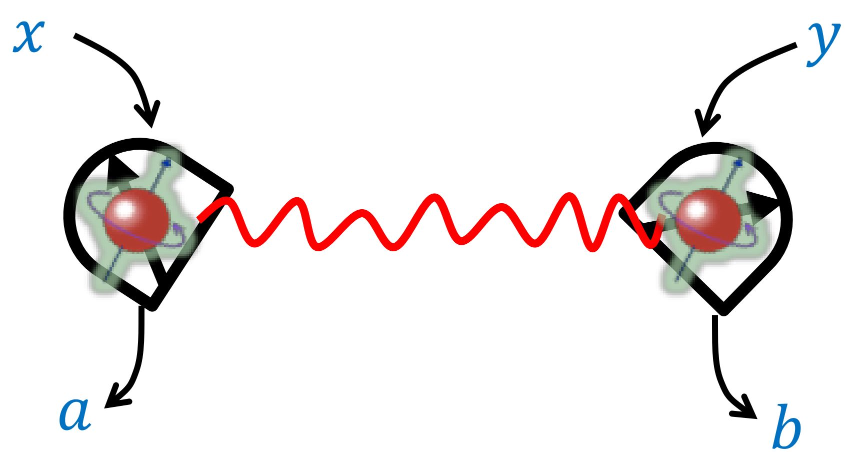
$C_{com}(n,k) = \{ \left(\langle\psi|P_a^x Q_b^y|\psi\rangle\right)_{abxy} :\ |\psi\rangle \in \mathcal{H}, [P_a^x, Q_b^y] = 0\}$

- Tsirelson's problem: **Is $\overline{C_\otimes(n,k)} = C_{com}(n,k)$ for all $n,k \geq 2$?**
- We give a negative answer: $\overline{C_\otimes(n,k)} \neq C_{com}(n,k)$ for some n,k
- Proof uses complexity theory to show that linear optimization over $C_\otimes$ is intractable
- Implies existence of Bell inequalities s.t. near-maximum violation requires quantum systems of *uncomputable* dimension
- Result has consequences for operator algebras (refutation of CEP/QWEP)
  → Further implications? Can we show existence of a non-hyperlinear/sofic group?
- Result implies a separation between two models for quantum correlations
  → Separating example is finite but *big*. Simplify, as was done in [DPP'17,C'19] for Slofstra's proof of non-closure of $C_\otimes$

Thank you



Nonlocal correlations



- Experimental data modeled as family of distributions $\{p(a,b|x,y)\} \in \mathbb{R}^{n^2k^2}$
  x, y: n possible measurement choices
  a, b: k possible measurement outcomes

- Bell 1964: some $\{p(a,b|x,y)\}$ have a model in QM but no classical explanation

- Tsirelson '80s: principled approach to study geometry of quantum correlations

- "Tsirelson's bound" (1980) places non-trivial limit on quantum correlations
  (non-trivial = stronger than no-signaling/relativity)