Hacking 2 048 RSA code with 8 100 qubits and a multimode memory with 2 hours storage time

Élie Gouzien and Nicolas Sangouard

December 3, 2020, GDR-IQFA'11 Colloquium

Institut de Physique Théorique, CEA Saclay

Quantum computing hardware

Possible harware for quantum computing
Superconducting qubits, trapped atoms/ions, optics, spins...

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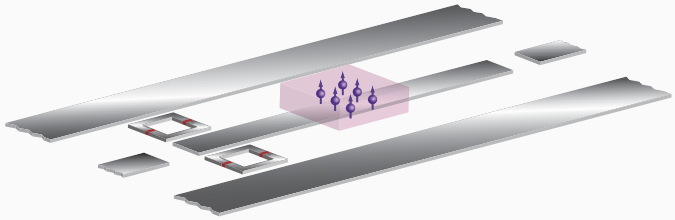
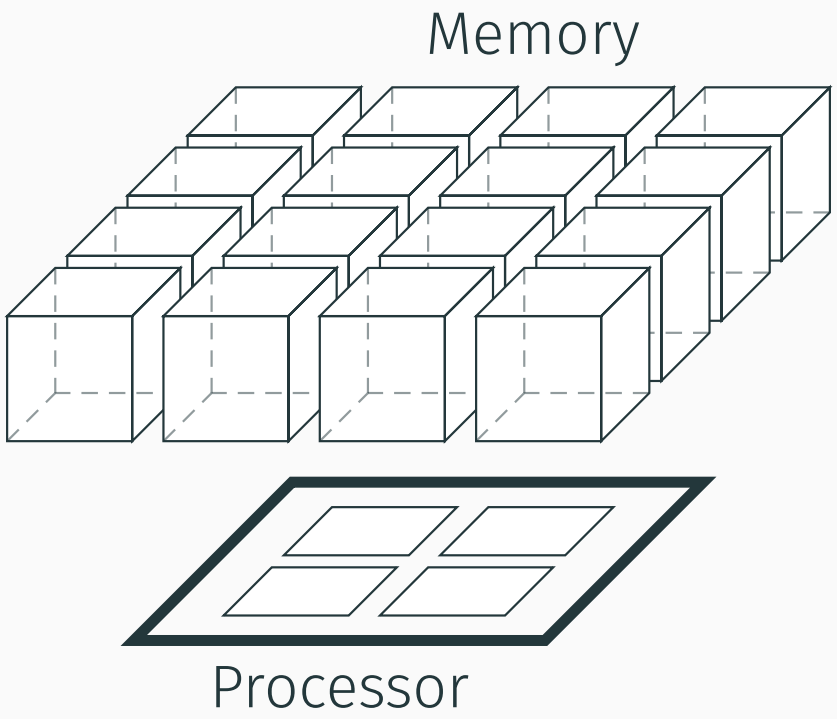
Superconducting processor + memory



Cécile Grezes et al. “Towards a spin-ensemble quantum memory for superconducting qubits”. *Comptes Rendus Physique* 17.7 (2016), pp. 693–704

Proposed architecture



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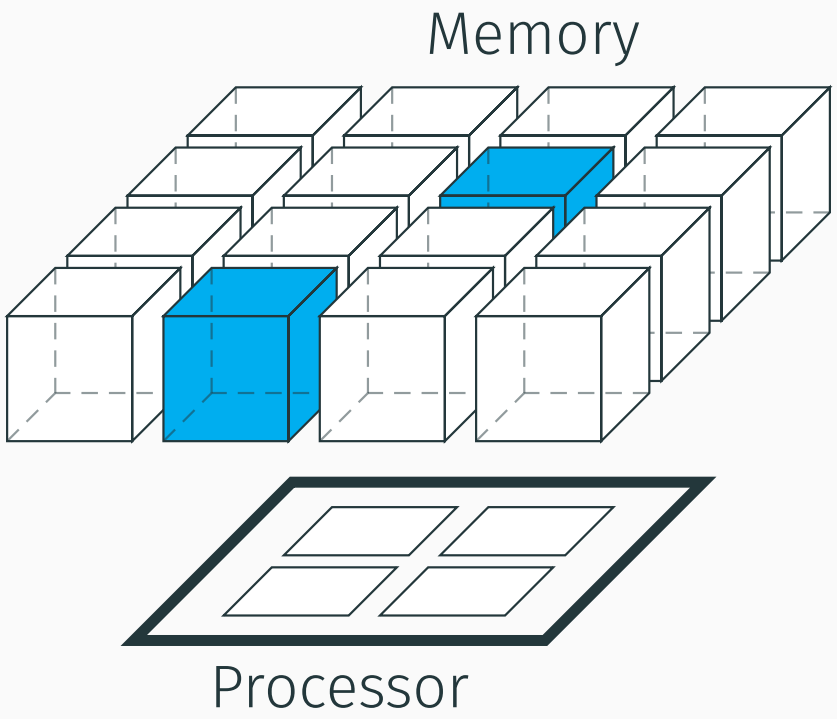

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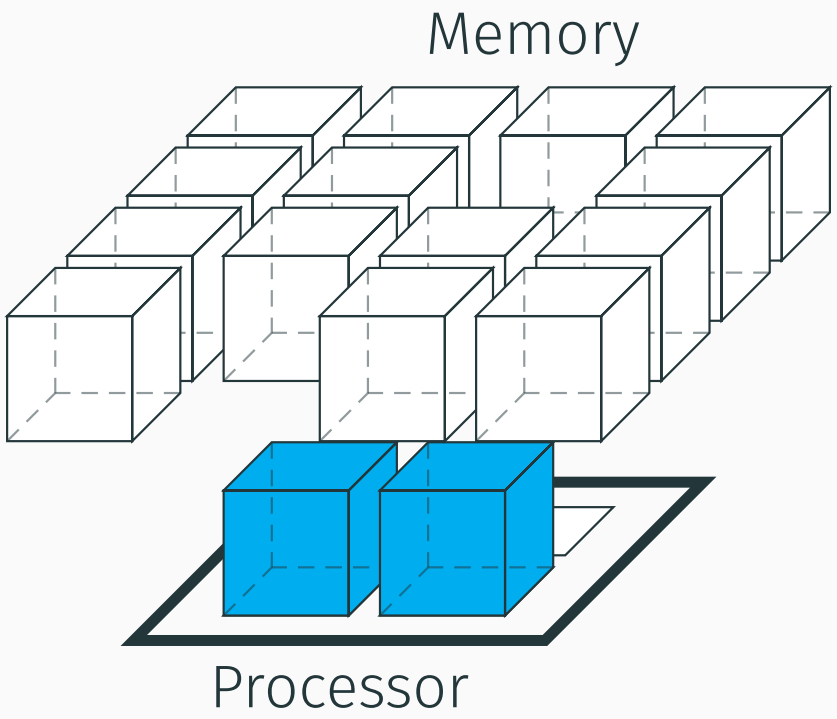

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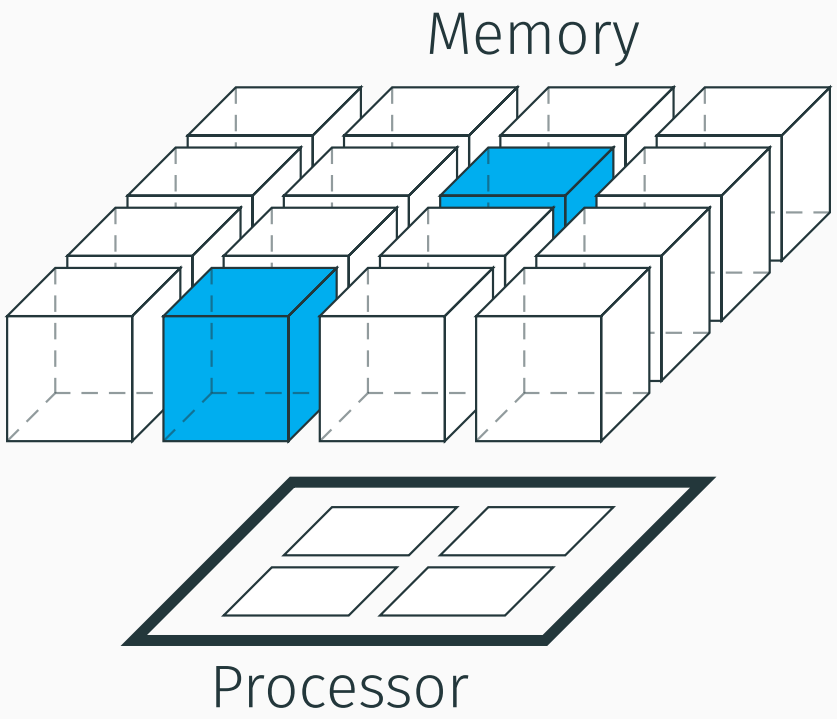

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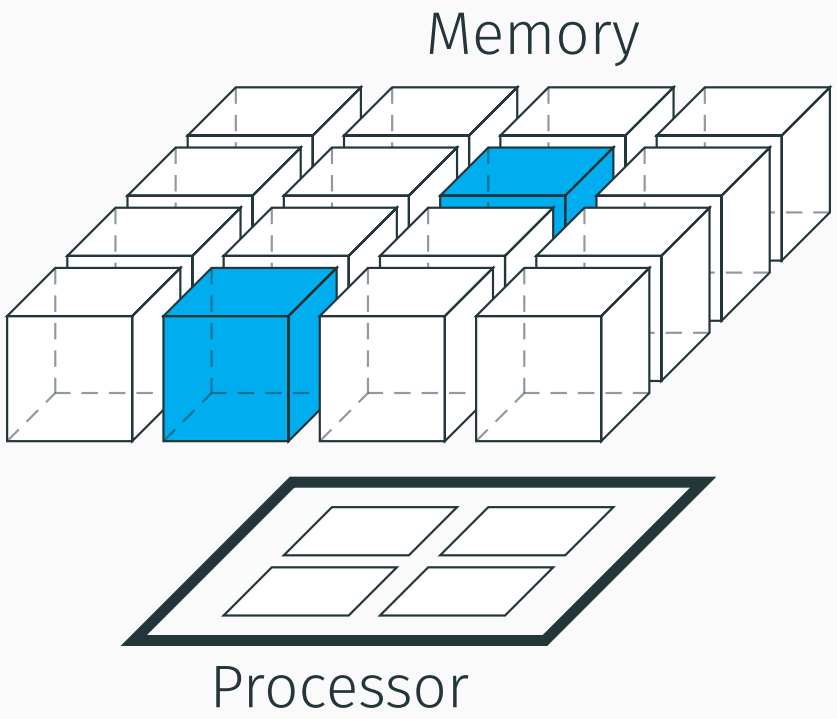

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- small processor
- better connectivity
- reduced decoherence for in-memory qubits

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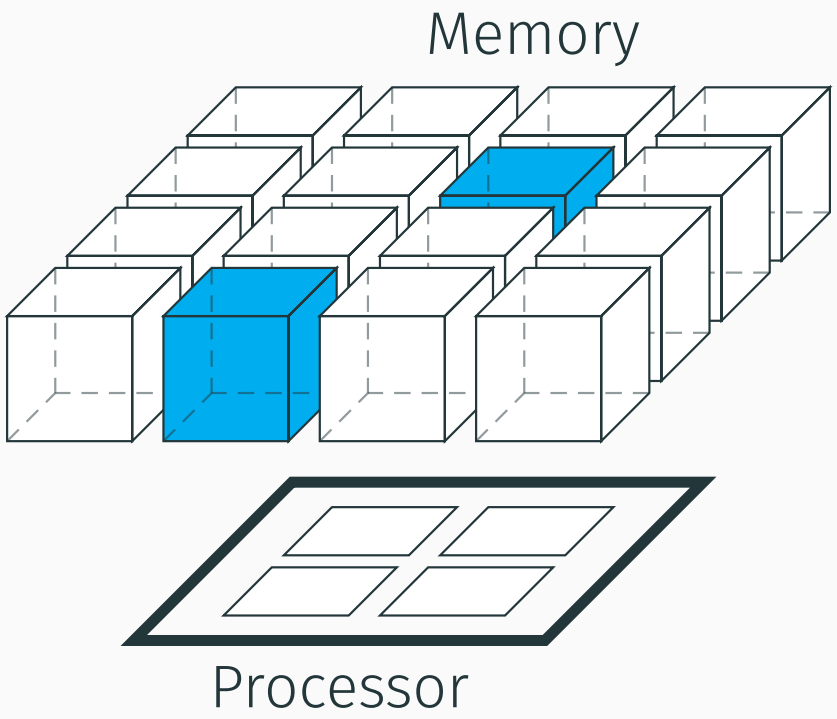

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What is quantitatively the advantage brought by this architecture?

RSA number prime factorization

Shor's algorithm (and variants)

- classically hard: no polynomial algorithm
- breaking RSA shows importance of quantum communications
- check the correct operation of the machine
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Factorization with 2D superconducting processor

Craig Gidney and Martin Ekerå. "How to factor 2048 bit RSA integers in 8 hours using 20 million noisy qubits". 2019

Factorization algorithm

Shor's algorithm

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$N = pq$, p and q prime numbers

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Run on quantum hardware

1. prepare $\frac{1}{\sqrt{2^{n_e}}} \sum_{x=0}^{2^{n_e}-1} |x\rangle$, n_e number of bits of the exponent
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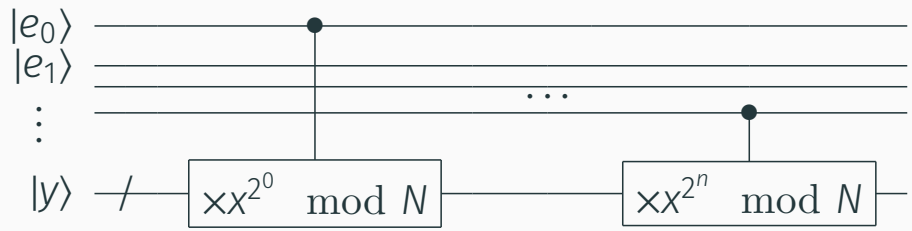
Standard implementation complexity

Exponentiation: $|e\rangle|y\rangle \mapsto |e\rangle\left|y \times x^e \mod N\right\rangle$

$$x^e = x^{\sum_i 2^i e_i} = \prod_i \left[x^{2^i}\right]^{e_i}$$



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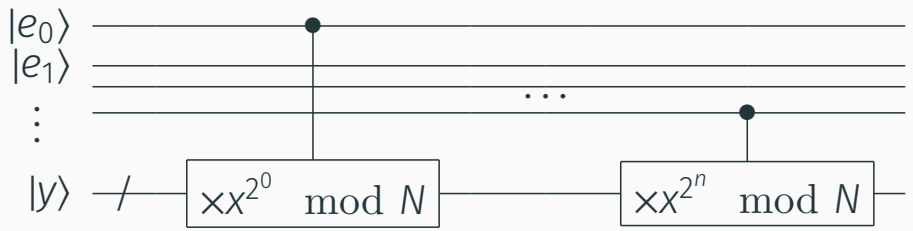

Multiplication: $|z\rangle|y\rangle \mapsto |z\rangle|y + x \times z \mod N\rangle$

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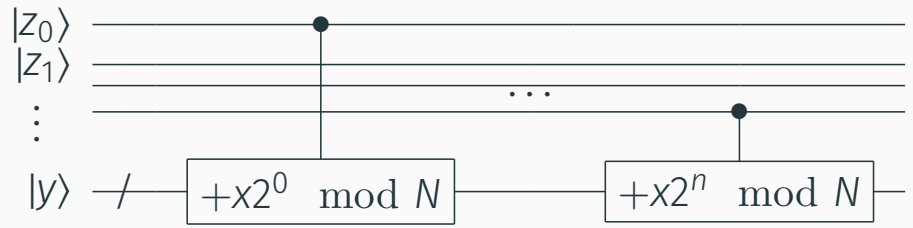

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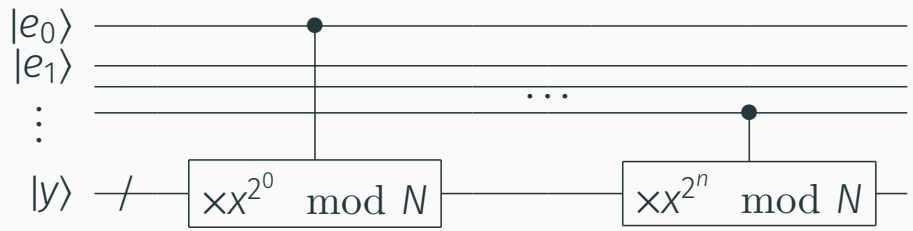

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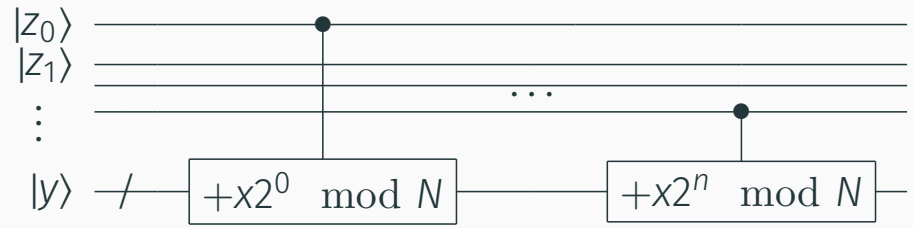

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$O(n)$ with carry-ripple addition
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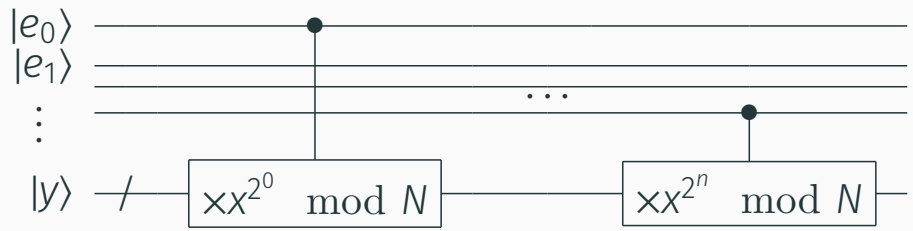

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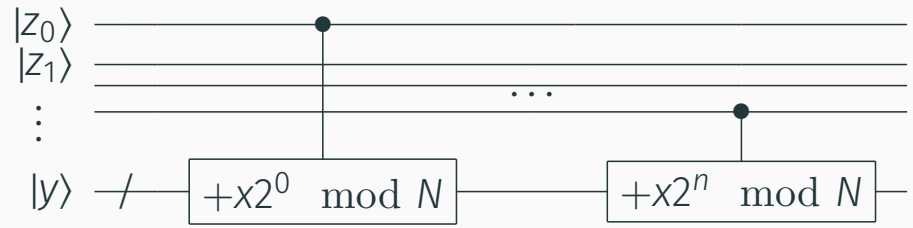

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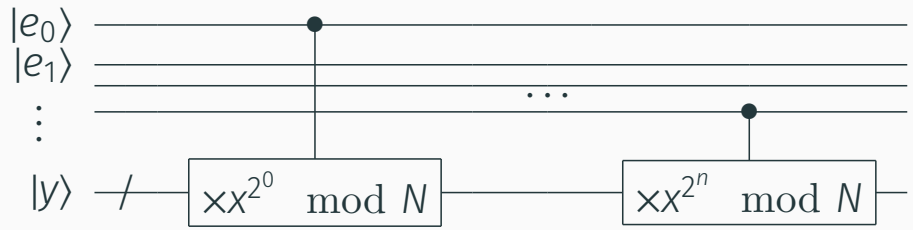

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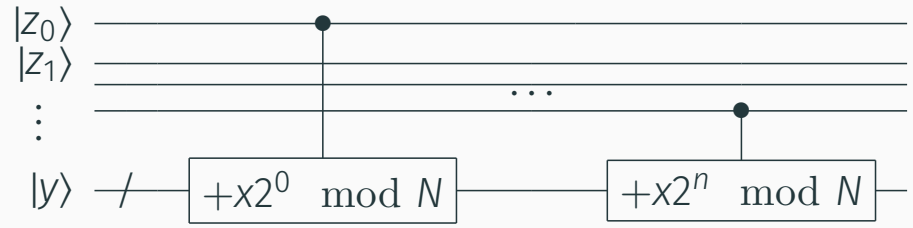

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Modular exponentiation: $O\left(n^3\right)$

Advanced algorithm complexity

Ekerå and Håstad's algorithm

Size of the exponent: $1.5n$ instead of $2n$ for Shor's algorithm

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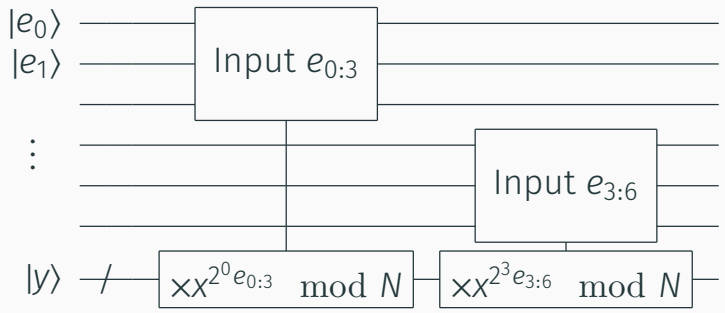

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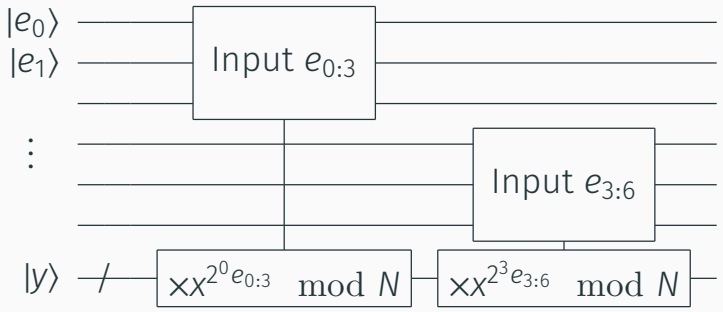

Modular addition through coset representation

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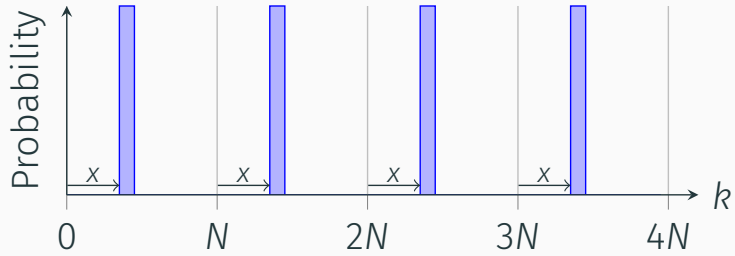
x represented by $\frac{1}{\sqrt{2^c}} \sum_{i=0}^{2^c-1} |x + iN\rangle$



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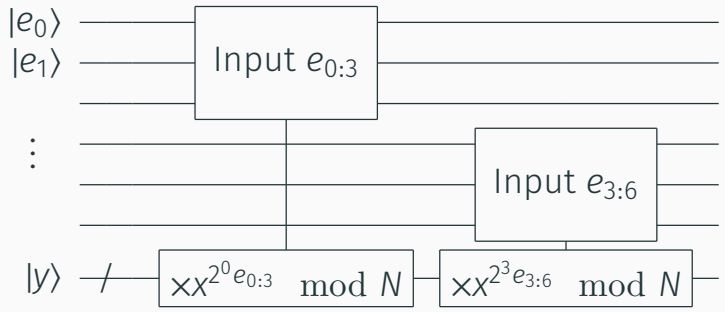

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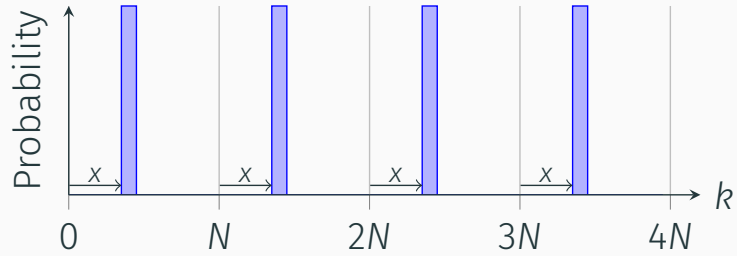

Complexity

parallel depth: $O\left(\frac{n^2\left(n+2^{2w}\right)}{w^2}\right)$, gates: $O\left(\frac{n^3 2^{2w}}{w^2}\right)$

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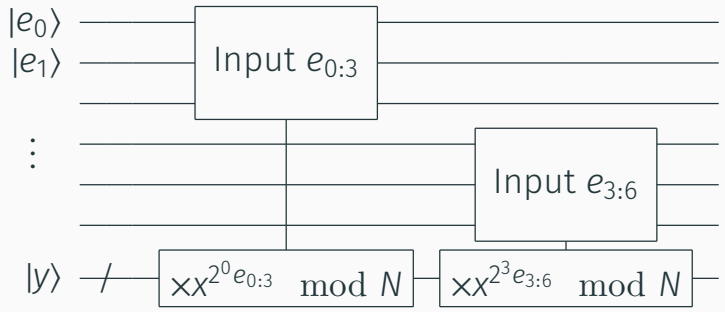

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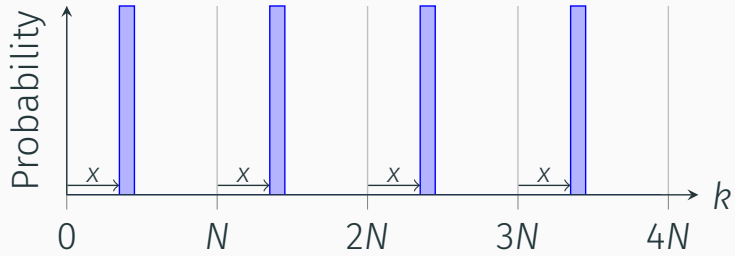

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$\Rightarrow O\left(n^3\right)$ but better pre-factors than basic algorithm

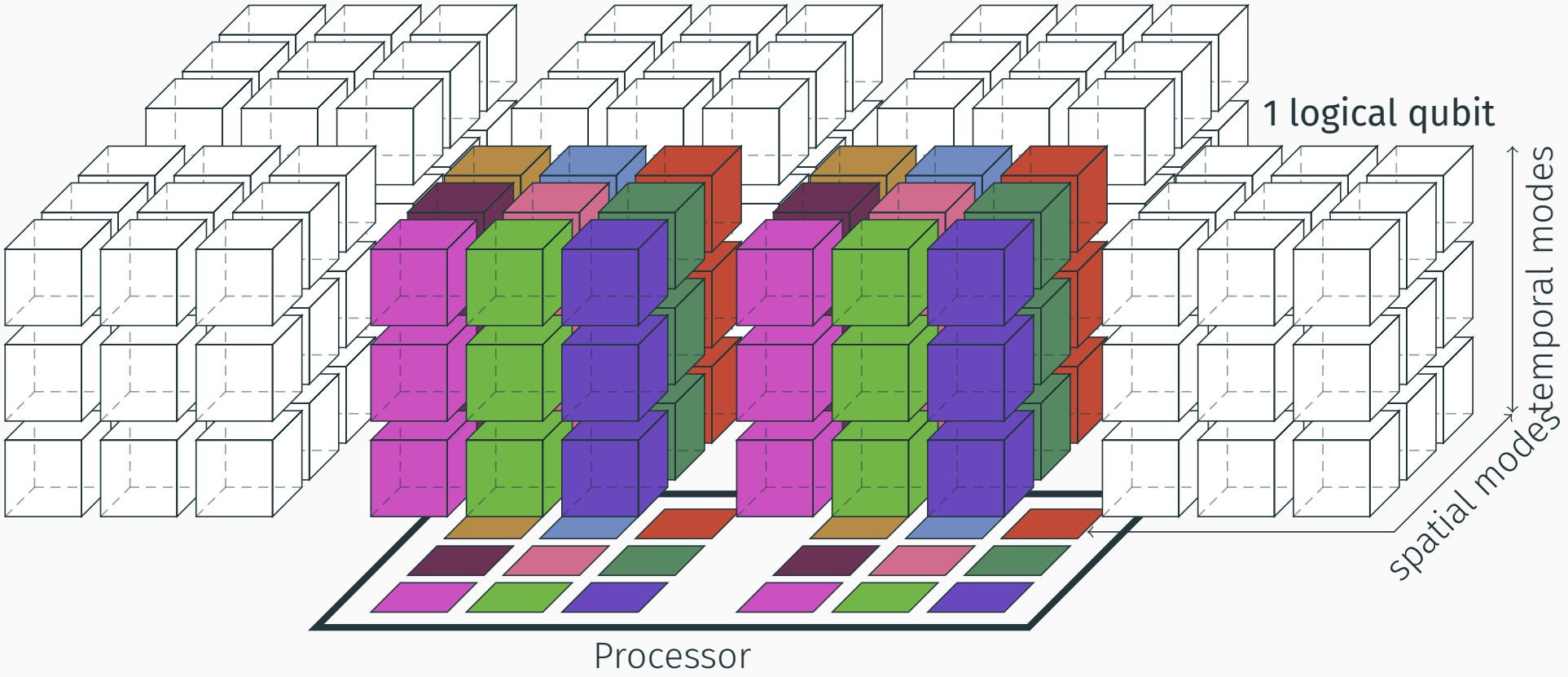
Error correction

3D gauge color codes

- H, CNOT and T gates transversal
  $\Rightarrow$ universal set
  $\Rightarrow$ no distillation/gate teleportation
- 3D code, 2D processor
  $\Longrightarrow$ 2D parallelism for stabilizers measurements
- single-shot error correction: no need to repeat stabilizers measurement
- Error threshold: 0.75 %

Héctor Bombín. "Gauge color codes: optimal transversal gates and gauge fixing in topological stabilizer codes". *New Journal of Physics* 17.8 (2015), p. 083002

How to measure syndromes



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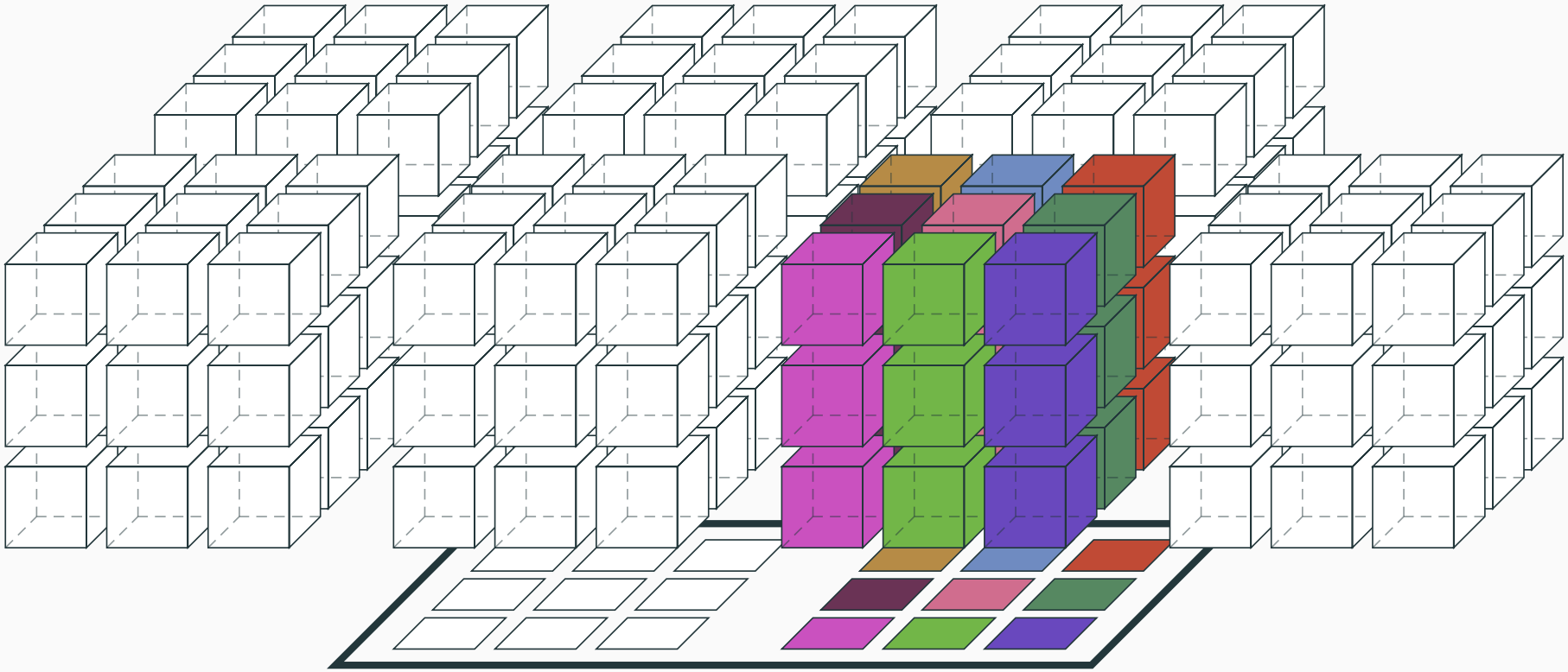

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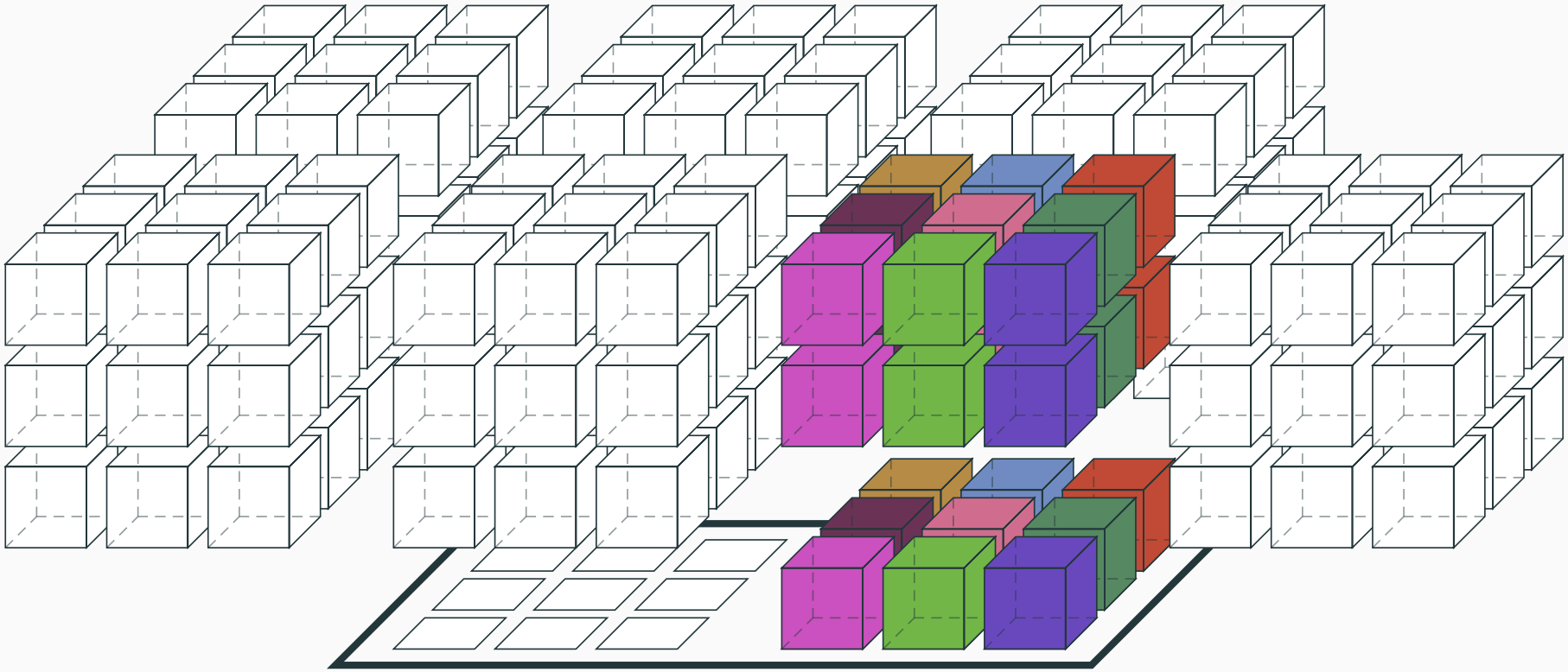

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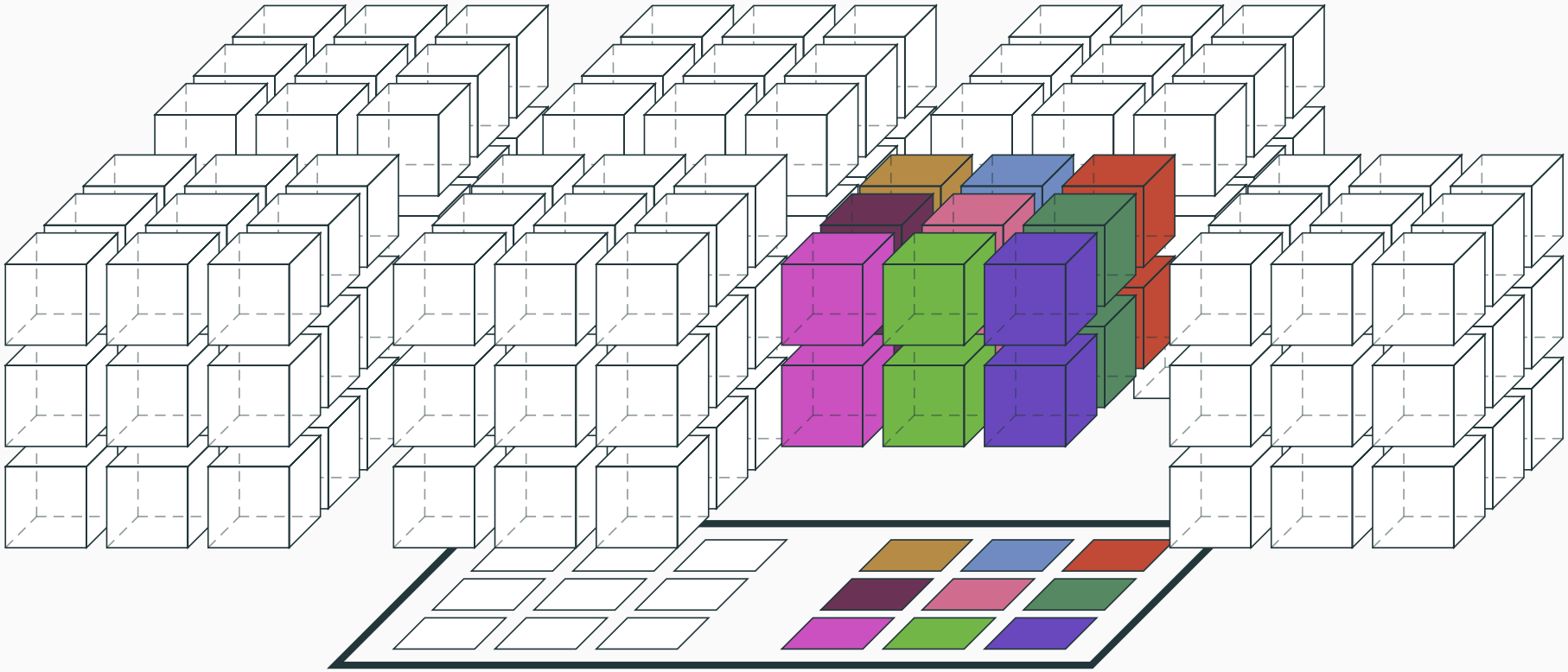

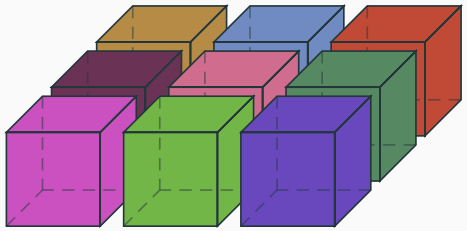

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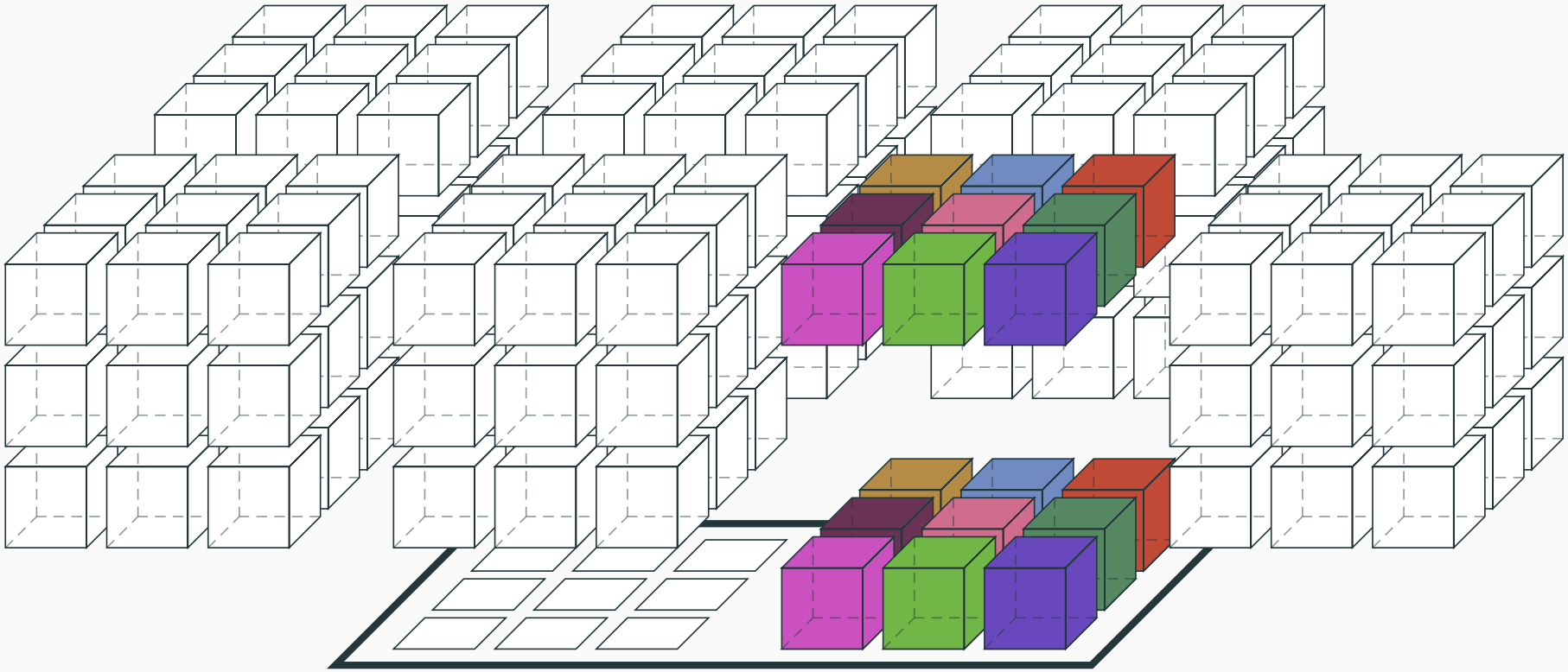

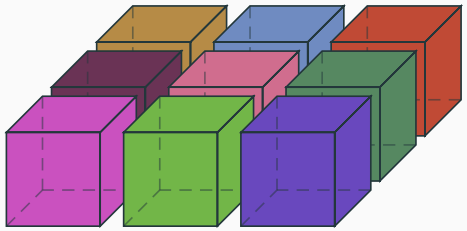

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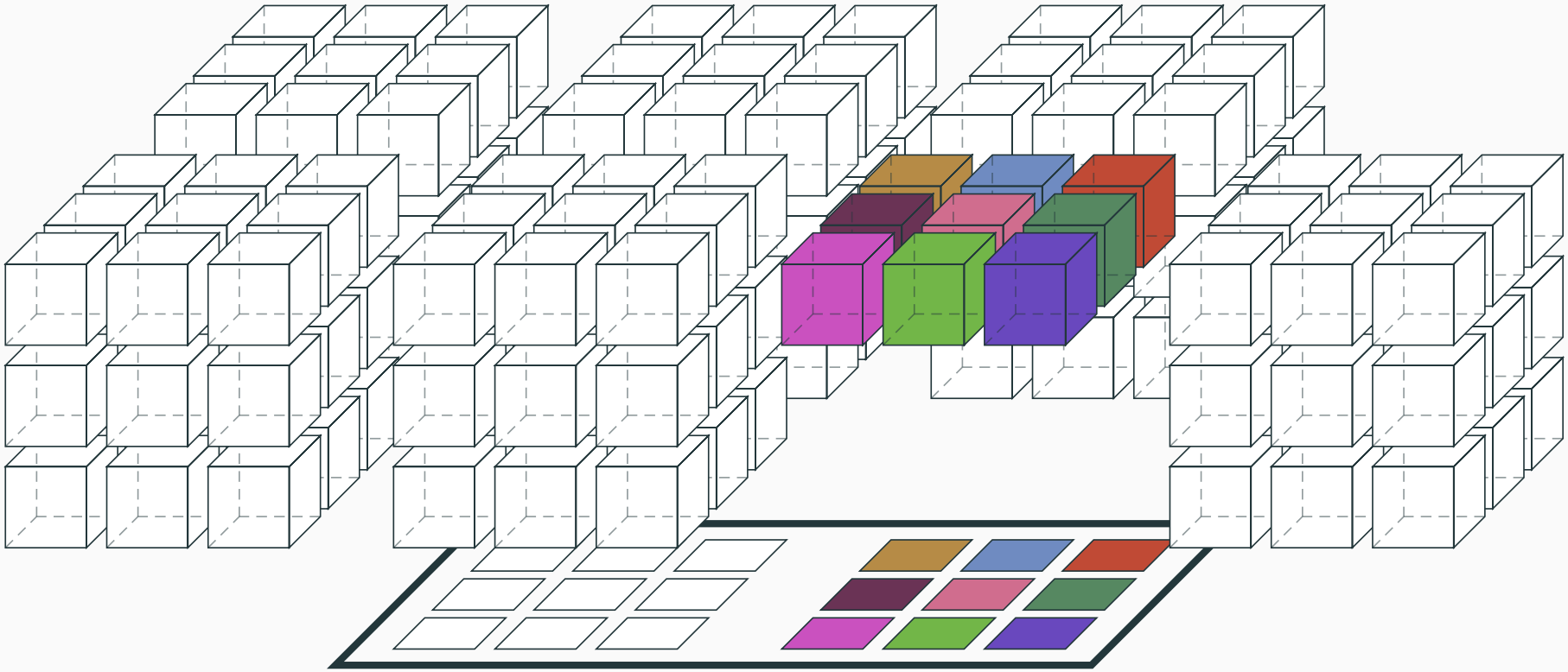

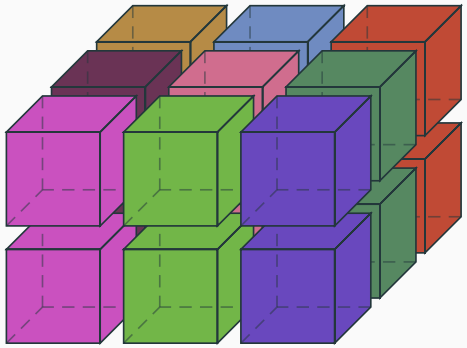

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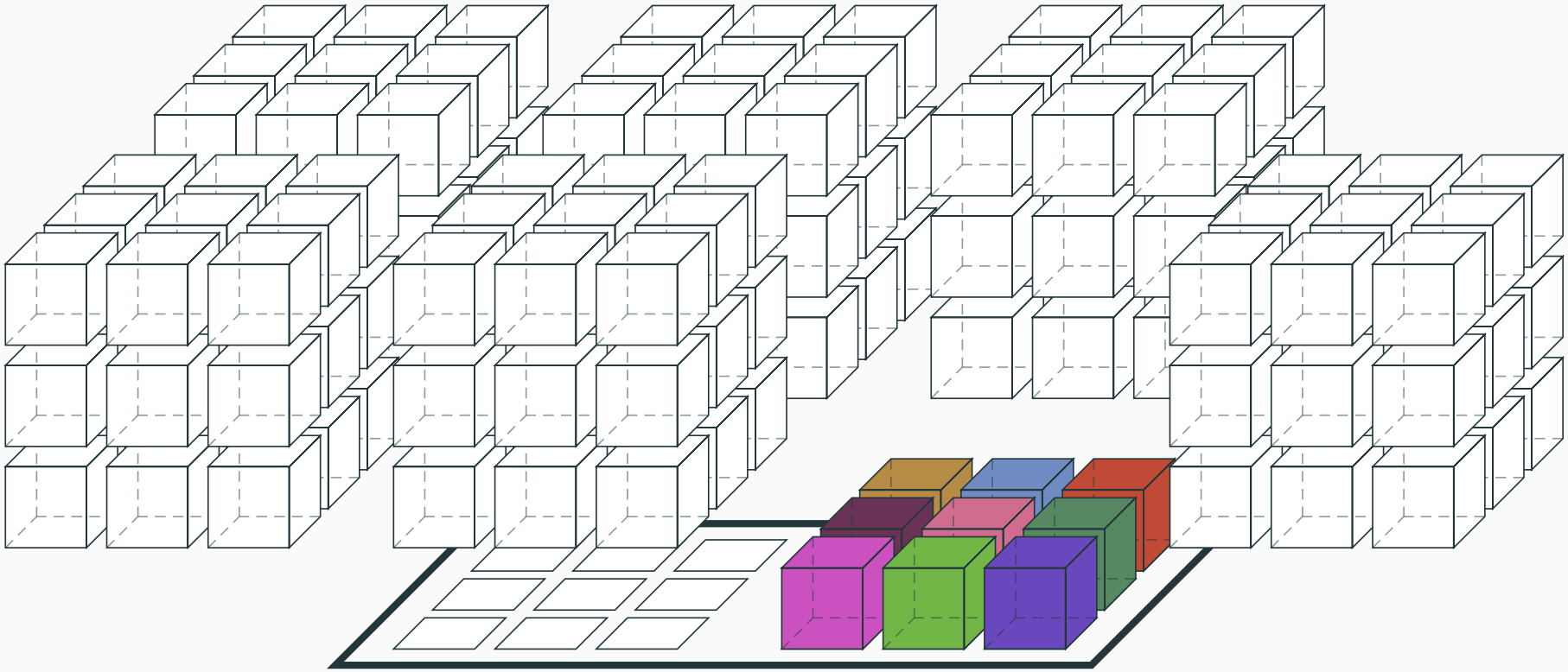
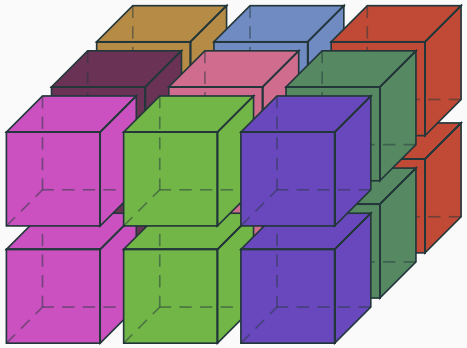
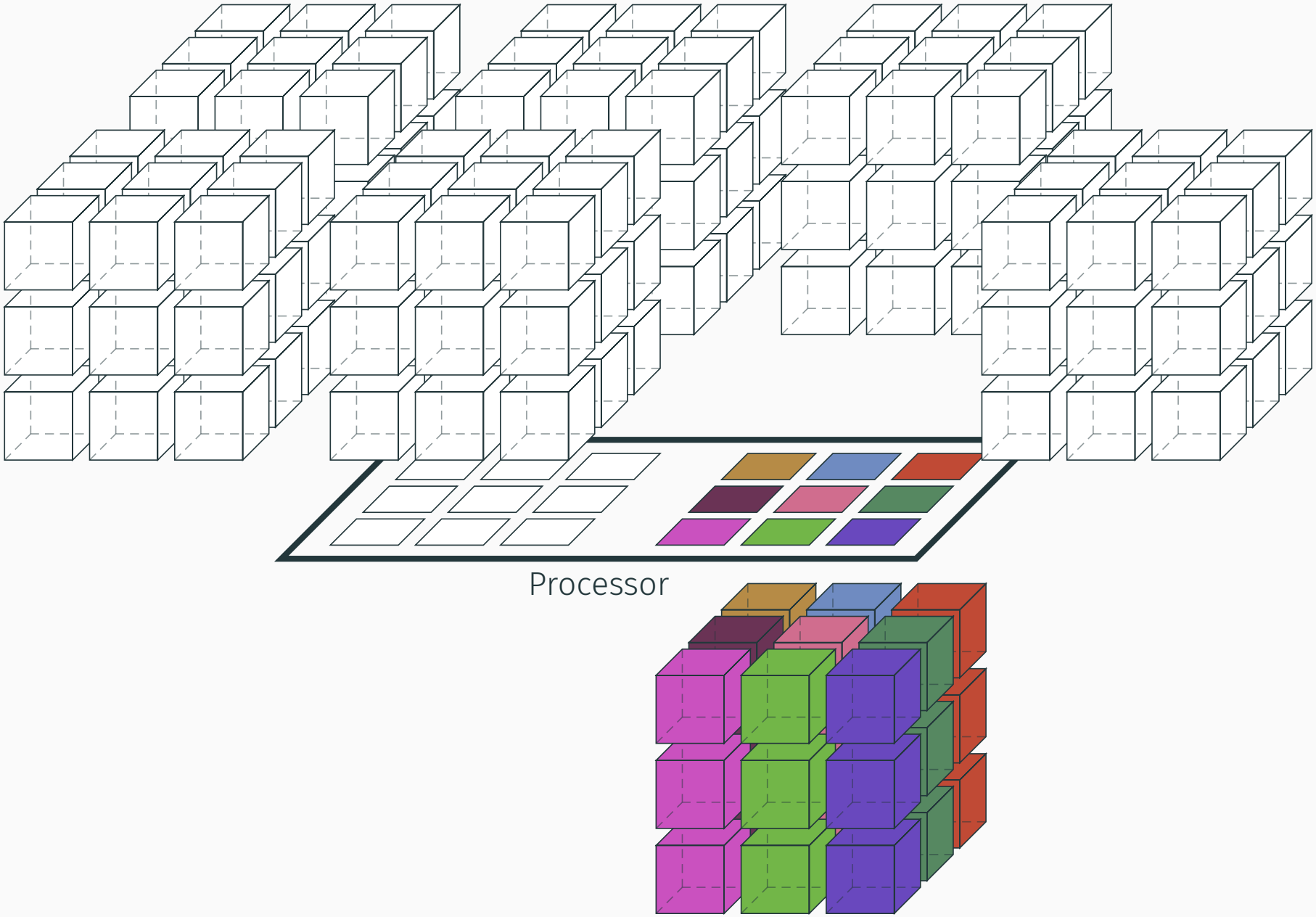


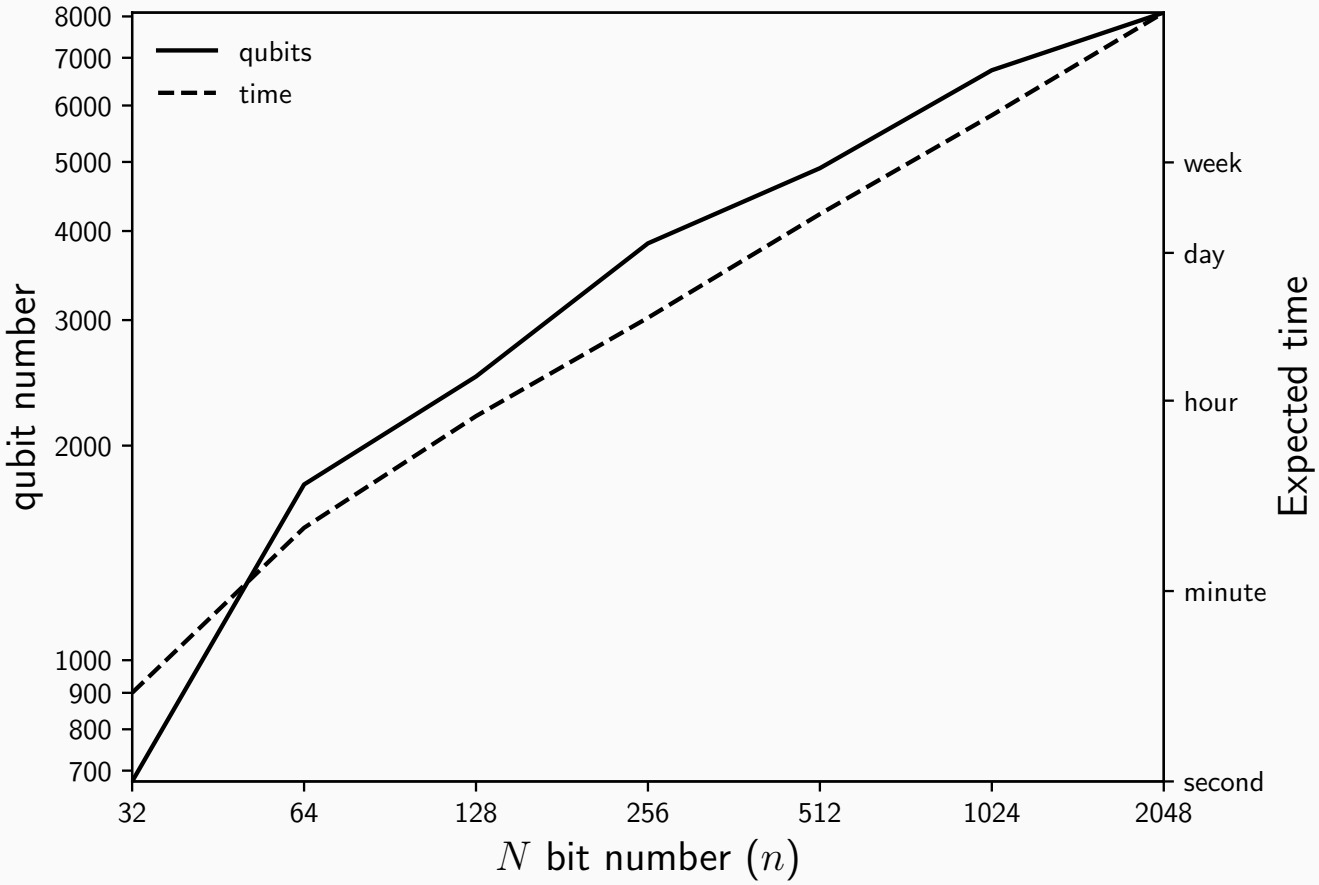


How to measure syndromes



Results

Resources required for factorization



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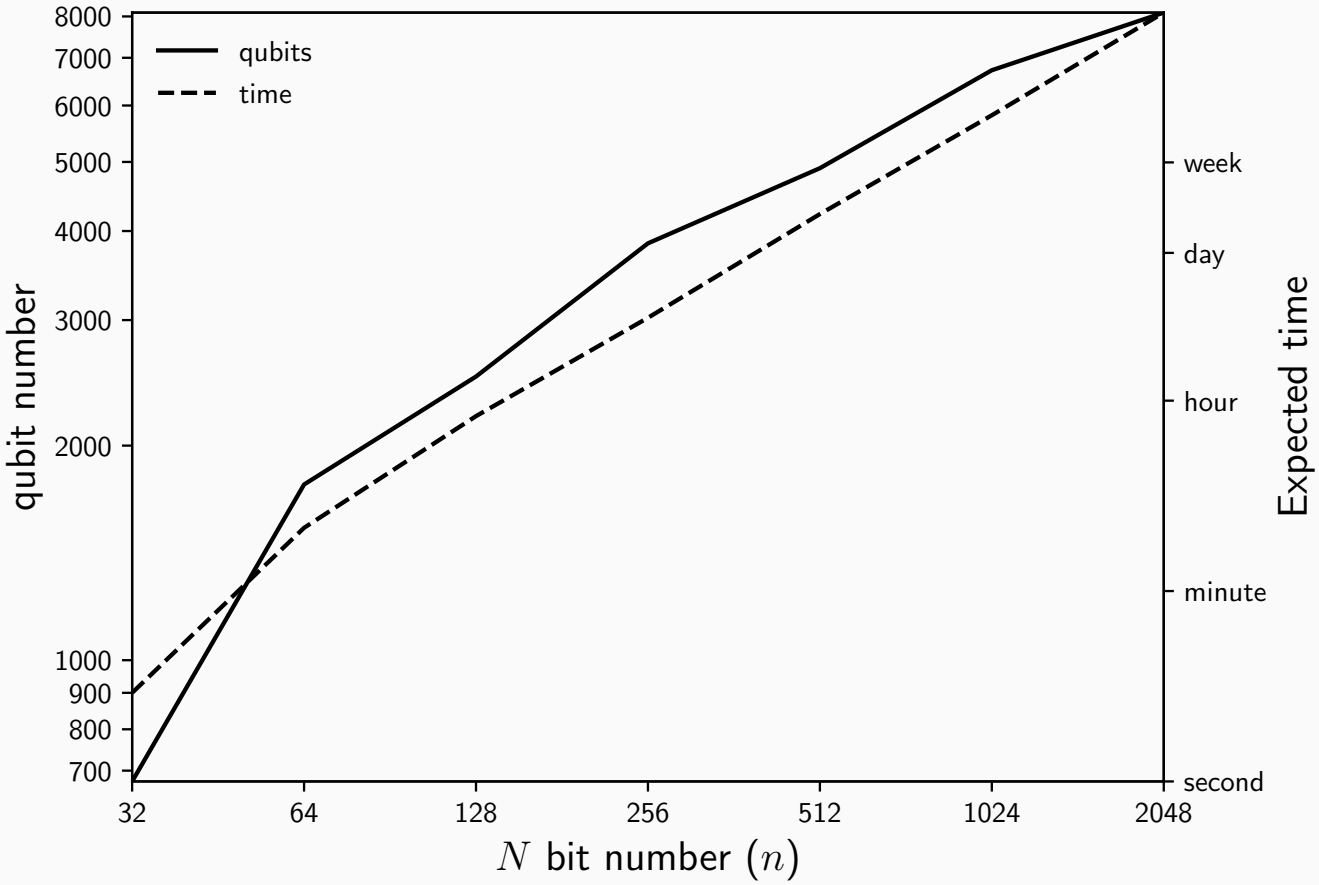

RSA 829 bits factorization
6 084 qubits, 10 days; maximum storage time: $\approx$ 15 min

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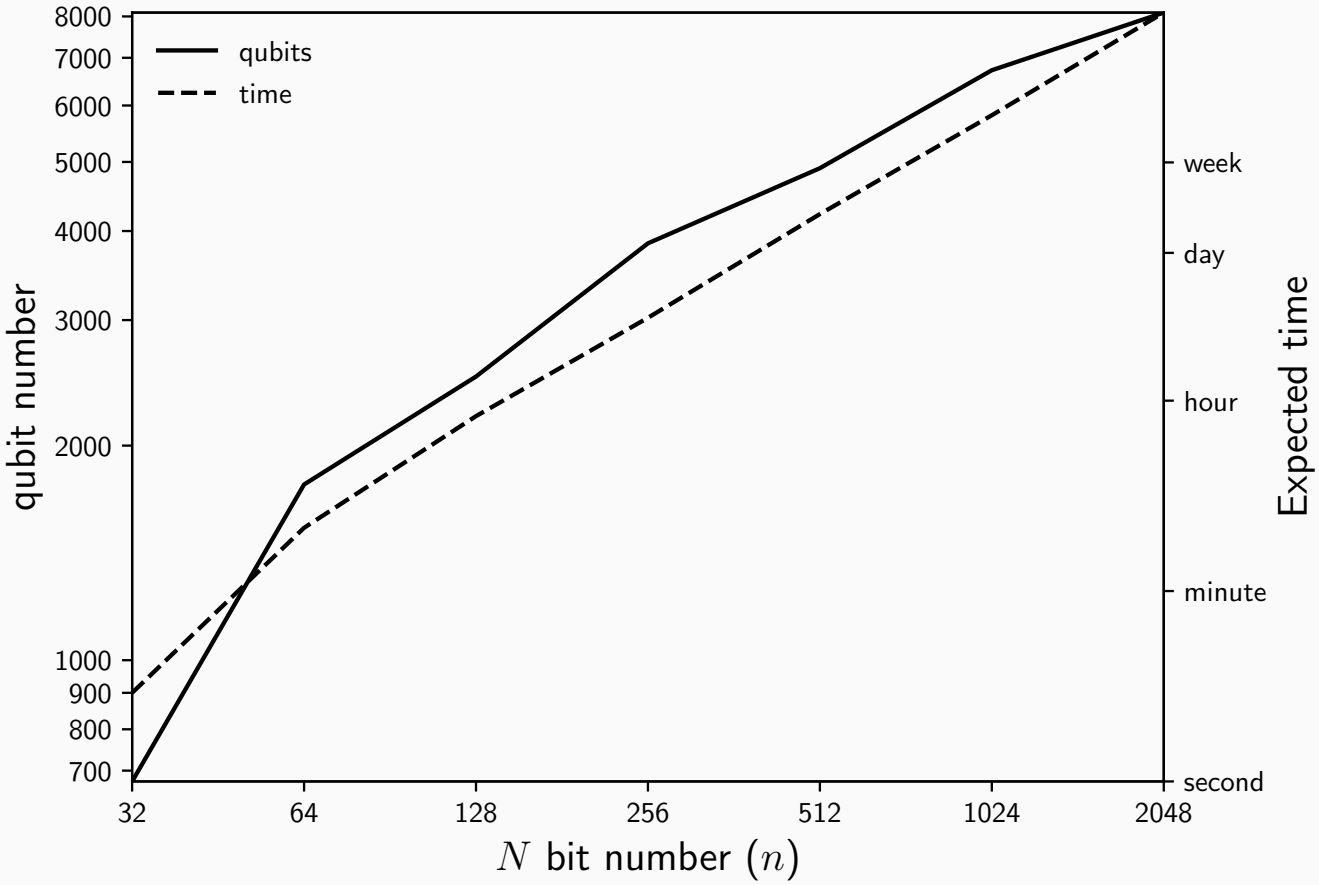

RSA 829 bits factorization
6 084 qubits, 10 days; maximum storage time: ≈ 15 min

RSA 2 048 bits factorization
8 100 qubits, 175 days; maximum storage time: ≈ 2 h

Comparison with 2D processor without memory

Only two logical qubit slices in the processor
9 277 logical qubits for RSA-2 048, only slices of 2 logical qubits in the superconducting processor.

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Reduced decoherence in the memory
Not really important because error-correction is exponentially efficient

Implementation and perspectives

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Implementation

- spin echo memory
- multimode cavity
- microwave photon router

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Additionnal improvement

- logical circuit more adapted to sequential computation
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Other problems

- optimization
- chemistry

The end

Thanks for your attention