Towards the ultimate precision limit in parameter estimation: Application to weak-value amplification and quantum speed limit

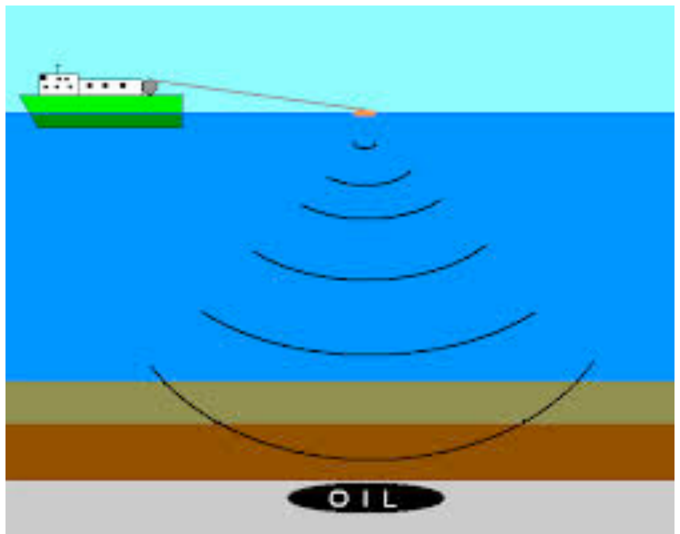
Luiz Davidovich

Instituto de Física - Universidade Federal do Rio de Janeiro

Based on work with Bruno M. Escher, Gabriel Bié Alves, Nicim Zagury, and Ruynet L. de Matos Filho

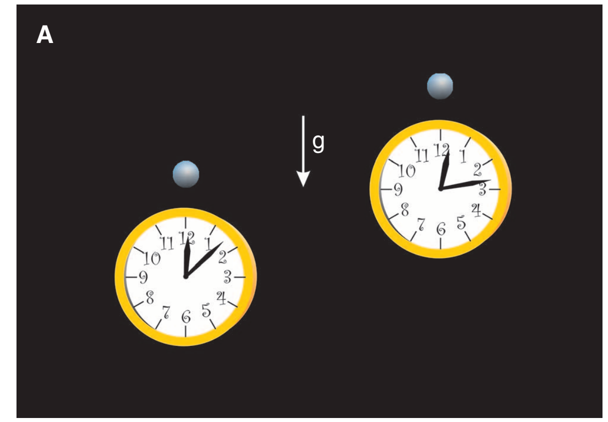
Parameter estimation

Depth of an oil well



Time duration of a process

Transition frequency



$$\Delta h = 33\,\mathrm{cm}$$

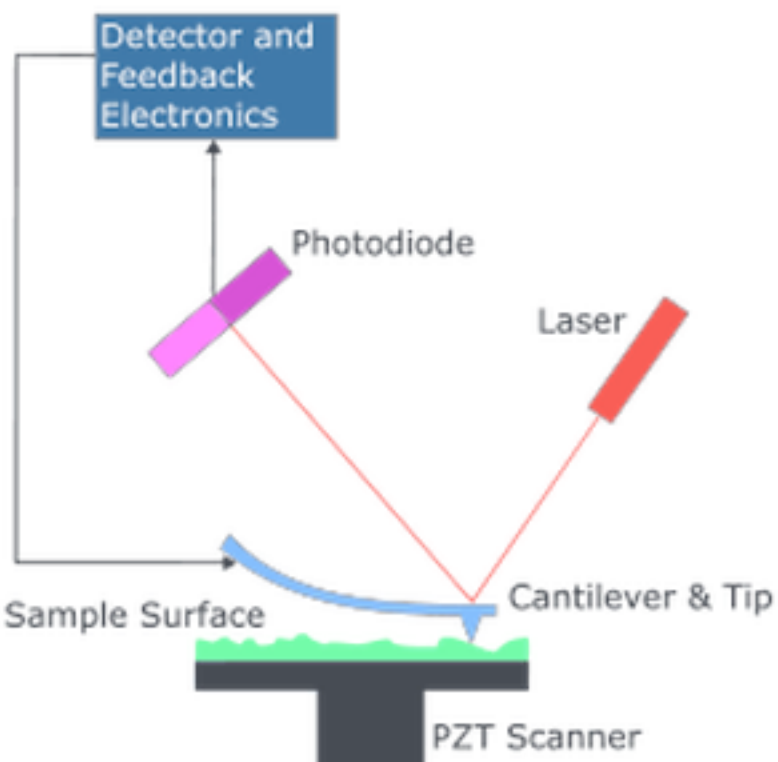
$$\frac{\Delta f}{f} = (4.1 \pm 1.6) \times 10^{-17}$$

Optical Clocks and Relativity

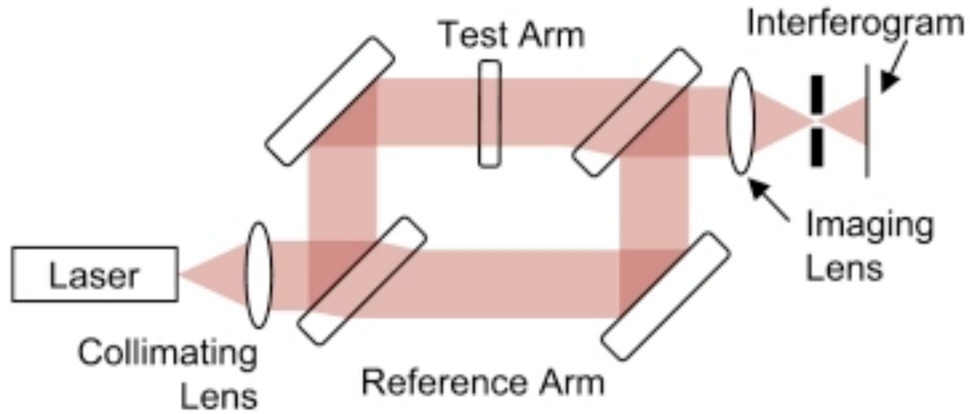
C. W. Chou,* D. B. Hume, T. Rosenband, D. J. Wineland

24 SEPTEMBER 2010 VOL 329 **SCIENCE**

Weak forces or small displacements



Phase displacements in interferometers



Laser Interferometer Gravitational Wave Observatory

nature photonics

LETTERS

PUBLISHED ONLINE: 21 JULY 2013 | DOI: 10.1038/NPHOTON.2013.177

Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light

The LIGO Scientific Collaboration*

Parameter estimation and uncertainty relations

What is the meaning of

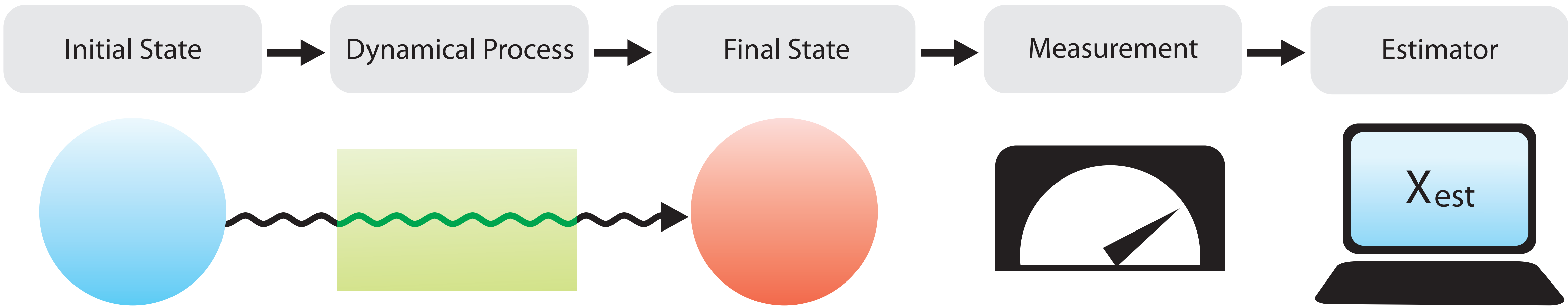
★ Time-energy uncertainty relation?

$$\Delta E \Delta T \geq \hbar / 2$$

★ Number-phase uncertainty relation?

$$\Delta N \Delta \phi \geq \hbar / 2$$

Parameter estimation in classical and quantum physics

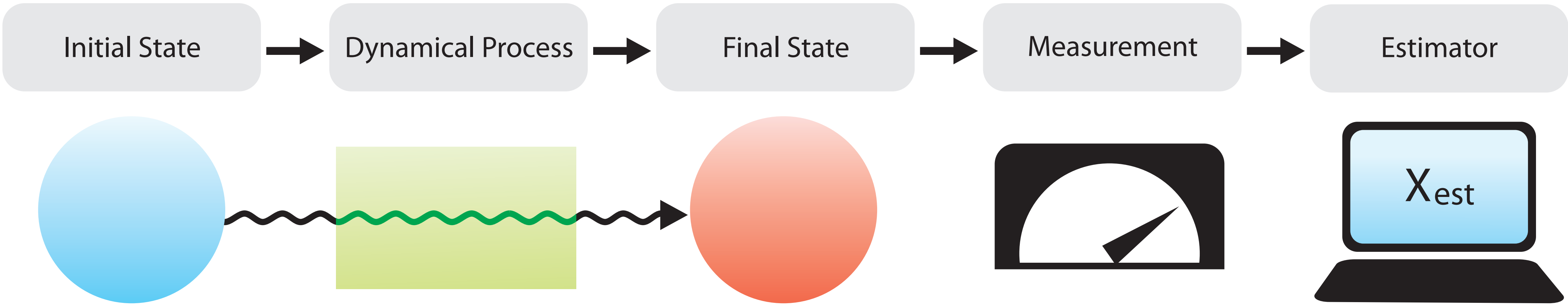


1. Prepare probe in suitable initial state
2. Send probe through process to be investigated
3. Choose suitable measurement
4. Associate each experimental result j with estimation

$$\delta X \equiv \sqrt{\left\langle \left[X_{\mathrm{est}}(j) - X \right]^2 \right\rangle_j \Big|_{X = X_{\mathrm{true}}}} \rightarrow \text{Merit quantifier}$$

$$\langle X_{\mathrm{est}} \rangle = X_{\mathrm{true}}\,,\; d\langle X_{\mathrm{est}} \rangle / dX = 1 \;\rightarrow\; \text{Unbiased estimator}$$

Quantum parameter estimation

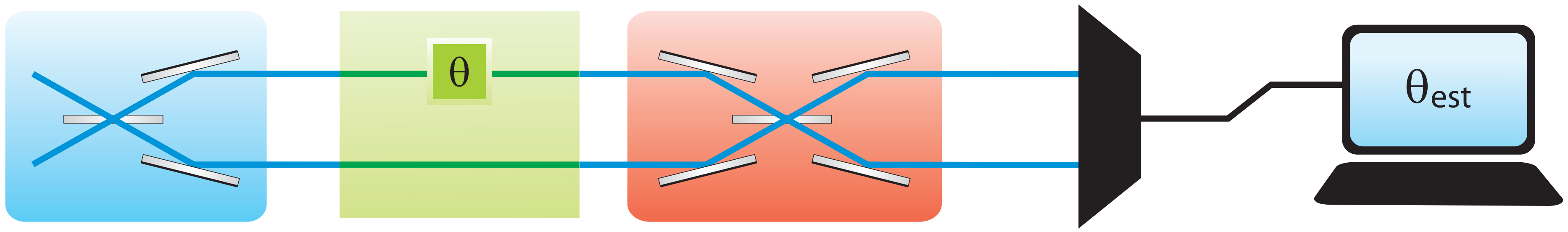


NEW POINTS TO CONSIDER

1. Precision in determination of parameter depends on the distinguishability between quantum states corresponding to nearby values of the parameter.
2. Measurement limited by quantum-mechanical restrictions
3. Is it possible to get better precision (for the same amount of resources) by using special quantum states?

RESOURCES: Number of atoms in atomic spectroscopy, number of photons in optical interferometry, average energy of a harmonic oscillator...

Example: Optical interferometry



Standard limit: $\delta\theta \approx \dfrac{1}{\sqrt{\langle n\rangle}}$

$$\left|\langle\alpha|\alpha e^{i\delta\theta}\rangle\right|^2 = \exp\left(-\left|\alpha\left(1-e^{i\delta\theta}\right)\right|^2\right)$$
$$\approx \exp\left[-\langle n\rangle(\delta\theta)^2\right] \Rightarrow \delta\theta \approx 1/\sqrt{\langle n\rangle}$$

Possible method to increase precision for the same average number of photons: Use NOON states [Bolinger et al., PRA 1996]

$$|\psi(N)\rangle = \left(|N,0\rangle + |0,N\rangle\right)/\sqrt{2} \to |\psi(N,\delta\theta)\rangle = \left(|N,0\rangle + e^{iN\delta\theta}|0,N\rangle\right)/\sqrt{2}, \quad (\langle n\rangle = N)$$

$$\boxed{\left|\langle\psi(N)|\psi(N,\delta\theta)\rangle\right|^2 = \cos^2(N\delta\theta/2) \Rightarrow \delta\theta \approx 1/N}$$ Heisenberg limit

Precision is better, for the same amount of resources (average number of photons)!

Cramér, Rao, and Fisher

H. Cramér

C. R. Rao

R.A. Fisher

1922, 1925

Fisher information

Cramér-Rao bound for unbiased estimators:

$$\delta X \geq 1/\sqrt{\nu F(X_{\text{true}})},\quad F(X) \equiv \int d\xi\; p(\xi \mid X)\left(\frac{d\,\ln\left[p(\xi \mid X)\right]}{dX}\right)^2$$

$\nu \rightarrow$ Number of repetitions of the experiment

$p(\xi \mid X) \rightarrow$ probability density of getting an experimental result ξ

(Average over all experimental results)

Asymptotically attainable when $\nu \rightarrow \infty$

Quantum Fisher Information

(Helstrom, Holevo, Braunstein and Caves)

$$\boxed{F\left(X;\{\hat{E}_\xi\}\right) \equiv \int d\xi \, p(\xi \mid X)\left(\frac{d \ln\left[p(\xi \mid X)\right]}{dX}\right)^2}$$

$$\boxed{p(\xi \mid X) = \mathrm{Tr}\left[\hat{\rho}(X)\hat{E}_\xi\right]}$$

$$\boxed{\int d\xi \hat{E}_\xi = \hat{1}}$$

POVM

This corresponds to a given quantum measurement. Ultimate lower bound for $\langle(\Delta X_{\mathrm{est}})^2\rangle$: optimize over all quantum measurements so that

$$\boxed{\mathscr{F}_Q(X) = \max_{\{E_\xi\}} F\left(X;\left\{E_\xi\right\}\right)}$$

Quantum Fisher Information

Geometrical interpretation

$$\boxed{\sqrt{\mathscr{F}_Q}/2 \rightarrow \text{ speed}}$$

Bures' Fidelity: $\Phi_B(\hat{\rho}_1,\hat{\rho}_2) \equiv \left(\mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}}\right)^2 = \left|\langle\psi_1|\psi_2\rangle\right|^2$ (pure states)

$$\Rightarrow \Phi_B\left[\hat{\rho}(X),\hat{\rho}(X+\delta X)\right] = 1-(\delta X/2)^2\mathscr{F}_Q\left[\hat{\rho}(X)\right] + O\left[(\delta X)^4\right]$$

Related to distinguishability between states

Quantum Fisher information for pure states

Initial state of the probe: $|\psi(0)\rangle$

Final X-dependent state: $|\psi(X)\rangle = \hat{U}(X)|\psi(0)\rangle$, $\hat{U}(X)$ unitary operator.

Then (Helstrom 1976):

$$\mathcal{F}_Q(X) = 4\langle(\Delta\hat{H})^2\rangle_0\,, \quad \langle(\Delta\hat{H})^2\rangle_0 \equiv \langle\psi(0)|\left[\hat{H}(X) - \langle\hat{H}(X)\rangle_0\right]^2|\psi(0)\rangle$$

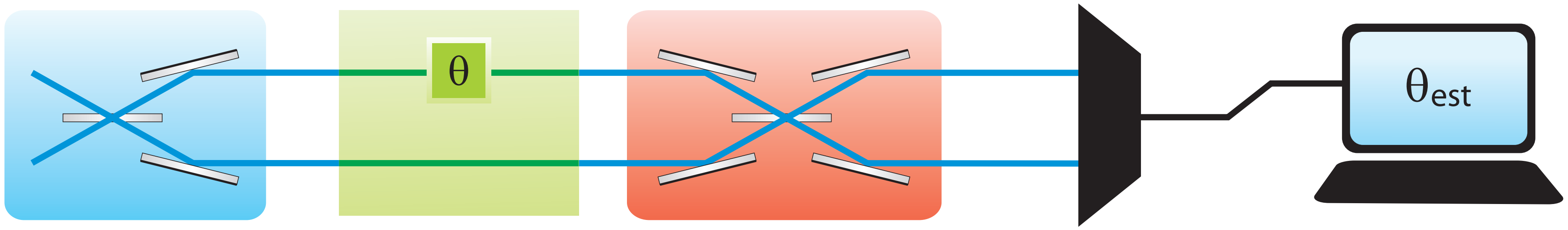
where

$$\hat{H}(X) \equiv i\frac{d\hat{U}^\dagger(X)}{dX}\hat{U}(X)$$

If $\hat{U}(X) = \exp(i\hat{O}X)$, $\hat{O}$ independent of X, then $\hat{H} = \hat{O}$

$$\delta x \geq 1/2\sqrt{\nu\left\langle\Delta\hat{H}^2\right\rangle}$$

Optical interferometry



$\hat{n} = \hat{a}^\dagger a \rightarrow$ Generator of phase displacements

$\Rightarrow \mathcal{F}_Q(\theta) = 4\langle(\Delta\hat{n})^2\rangle_0$ where $\langle(\Delta\hat{n})^2\rangle_0$ is the photon-number variance in the upper arm.

$$\Rightarrow \delta\theta \geq \frac{1}{2\sqrt{\langle(\Delta\hat{n})^2\rangle}} \qquad (\nu = 1)$$

Standard limit: coherent states

(Poissonian distribution of photons)

$$\boxed{\mathcal{F}_Q(\theta) = 4\langle(\Delta\hat{n})^2\rangle_0 = 4\langle\hat{n}\rangle \Rightarrow \delta\theta \geq \frac{1}{2\sqrt{\langle n\rangle}}}$$

Increasing the precision: maximize variance with NOON states:

$$|\psi(N)\rangle = (|N,0\rangle + |0,N\rangle)/\sqrt{2}$$

$$\boxed{\langle(\Delta\hat{n})^2\rangle_0 = \frac{N^2}{4} \Rightarrow \delta\theta \geq \frac{1}{N}}$$

Precision is better, for the same amount of resources.

Quantum metrology and weak-value amplification

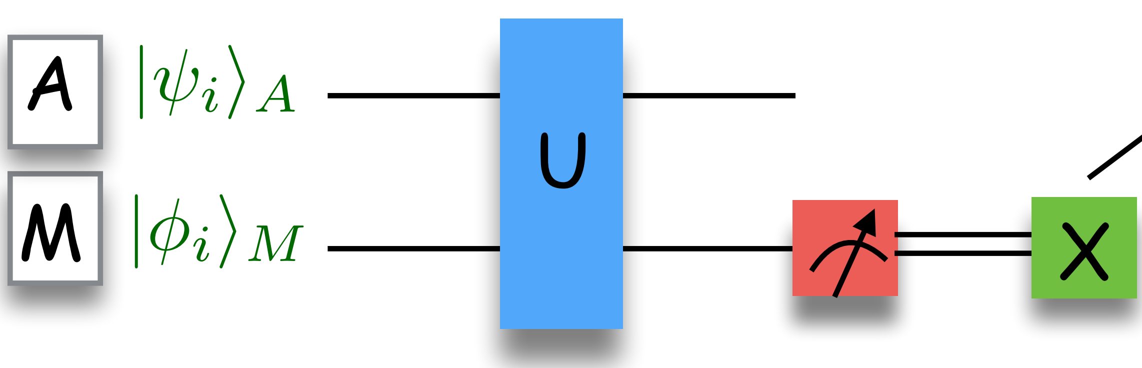
Usual framework: Start with Von Neumann measurement scheme

$$\hat{H}_I(t) = \hbar g \delta(t - t_0)\hat{A} \otimes \hat{M} \;\Rightarrow \hat{U}(g) = \exp(-ig\hat{A} \otimes \hat{M})$$ Free-evolution neglected

$\hat{A} \rightarrow$ System observable (assume discrete non-degenerate spectrum: $\hat{A}|a_i\rangle = a_i|a_i\rangle$)

$\hat{M} \rightarrow$ Meter observable (assume continuous spectrum)

Initial state of A+M: $|\Psi_i\rangle = |\psi_i\rangle_A \otimes |\phi_i\rangle_M$

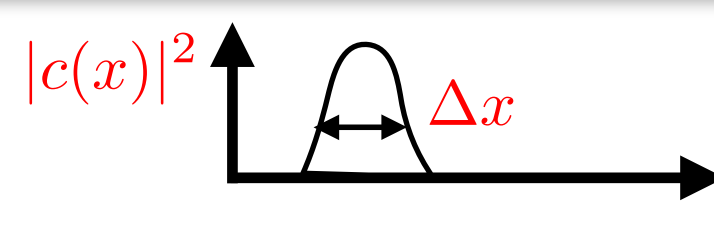


Value of pointer observable - canonically conjugate to $\hat{M}$

Strong (standard) measurement: $|g|\delta a \gg \Delta x$

Weak measurement: $|g|\delta a \lesssim \Delta x$

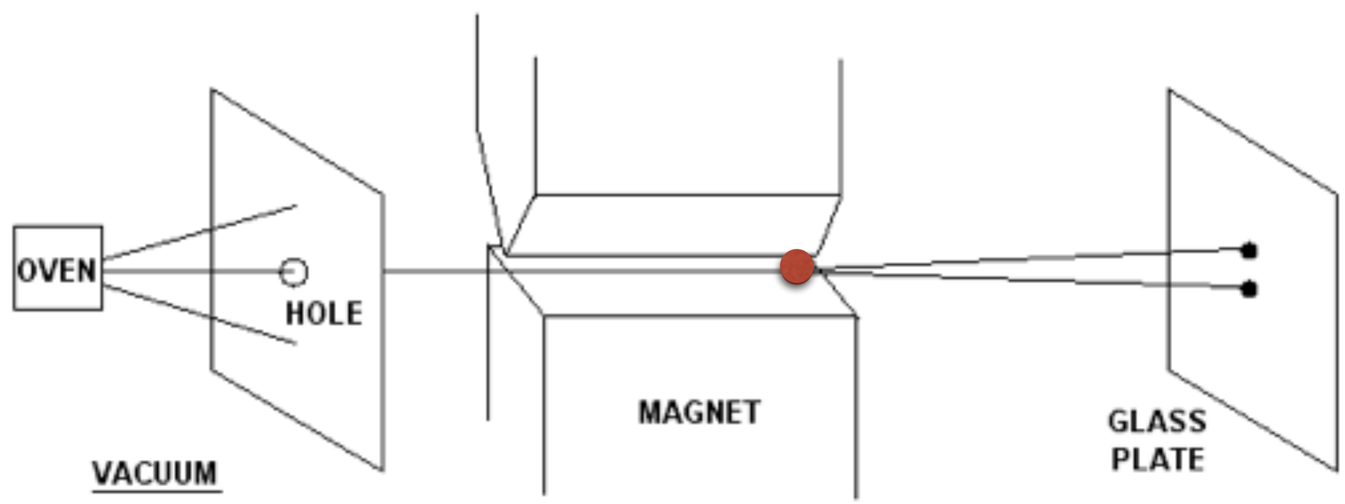
Example: If $\hat{M}$ = momentum, $\hat{X}$ = position



$$|\psi_i\rangle_A = \sum_i c_i|a_i\rangle\,, \;\; |\phi_i\rangle_M = \int dx\, c(x)|x\rangle$$

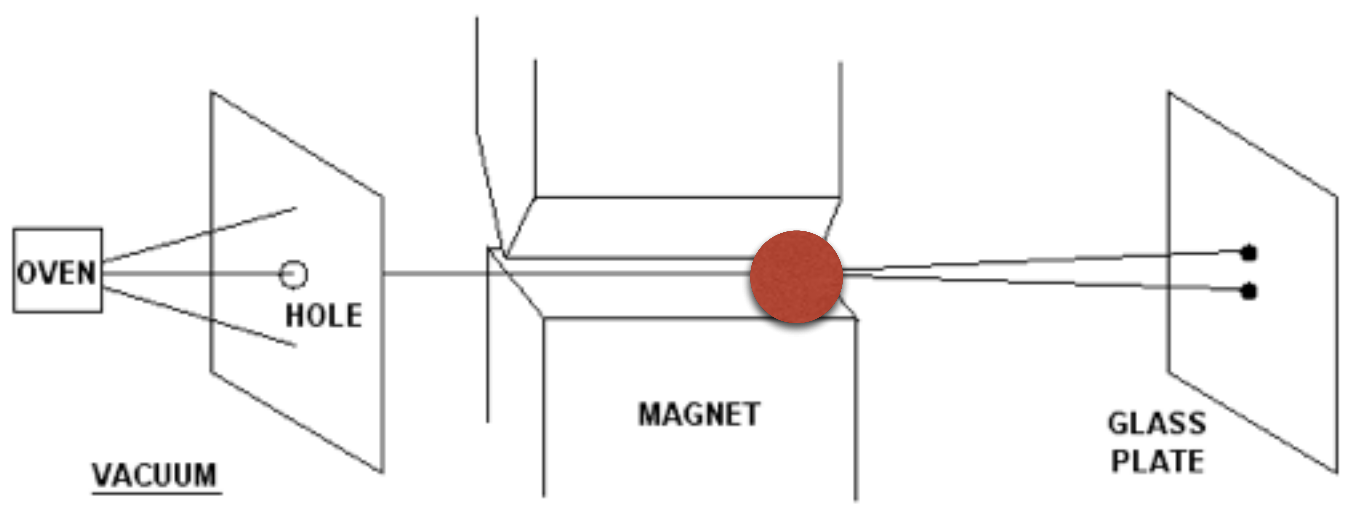
$$\Rightarrow |\Psi_f\rangle = \exp(-ig\hat{A} \otimes \hat{p})|\psi_i\rangle_A \otimes |\phi_i\rangle_M = \sum_i c_i|a_i\rangle \otimes \int dx\, c(x)|x - ga_i\rangle_M$$

Quantum metrology and weak-value amplification



The experiment of Stern and Gerlach

Strong

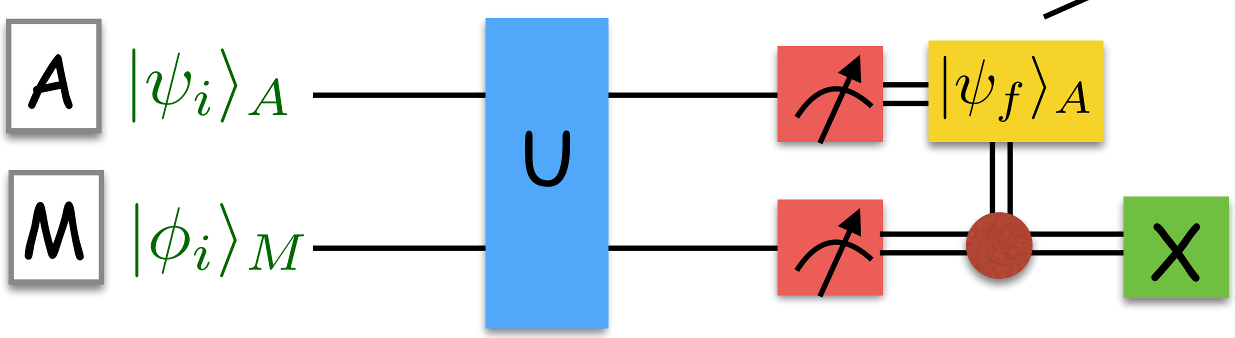


The experiment of Stern and Gerlach

Weak

Pre- and post-selected measurements

$$p_f(g) = |\langle\psi_f|\hat{U}(g)|\Psi_i\rangle|^2 \qquad |\Psi_i\rangle = |\psi_i\rangle_A \otimes |\phi_i\rangle_M$$



Measurement of X is conditioned on measurement of A in state $|\psi_f\rangle_A$

Unnormalized meter state after post-selection:

$$\begin{aligned}
&|\phi_f(g)\rangle_M = {}_A\langle\psi_f|\exp(-ig\hat{A}\otimes\hat{M})|\psi_i\rangle_A \otimes |\phi_i\rangle_M \\
&\approx {}_A\langle\psi_f|1 - ig\hat{A}\otimes\hat{M}|\psi_i\rangle_A \otimes |\phi_i\rangle_M \\
&= {}_A\langle\psi_f|\psi_i\rangle_A(1 - igA_w\hat{M})|\phi_i\rangle_M
\end{aligned}$$

$$\boxed{A_w = \frac{{}_A\langle\psi_f|\hat{A}|\psi_i\rangle_A}{{}_A\langle\psi_f|\psi_i\rangle_A} \rightarrow \text{Weak value}}$$

Two small parameters: g and δ

Could be much larger than $\langle\hat{A}\rangle$, by choosing $\delta = {}_A\langle\psi_f|\psi_i\rangle_A$ sufficiently small

Must have, however, $|gA_w|\Delta M \ll 1$ $\quad \Delta M \rightarrow$ width of $|\phi_i\rangle_M$

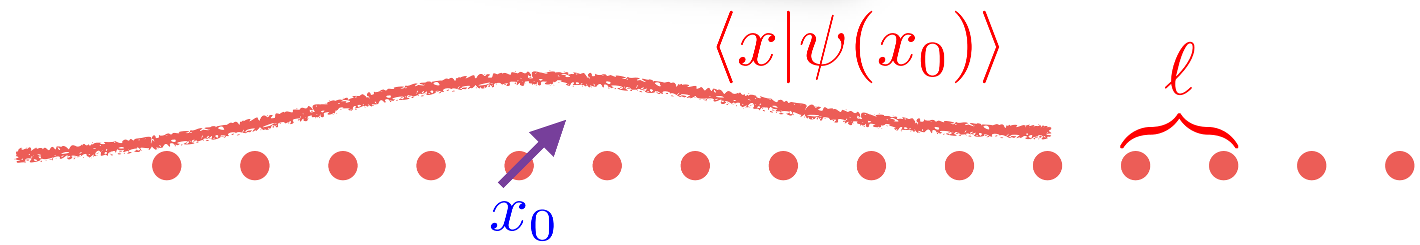
Example: Quantum version of random walks

PHYSICAL REVIEW A VOLUME 48, NUMBER 2 AUGUST 1993

Quantum random walks

Y. Aharonov,* L. Davidovich,† and N. Zagury†

$$\hat{U} = \exp(-i\hat{S}_z\hat{P}\ell/\hbar)$$



Initial state $|\Psi\rangle = |\psi(x_0)\rangle(c_\uparrow|\uparrow\rangle + c_\downarrow|\downarrow\rangle)$ $c_\uparrow,\, c_\downarrow$ real

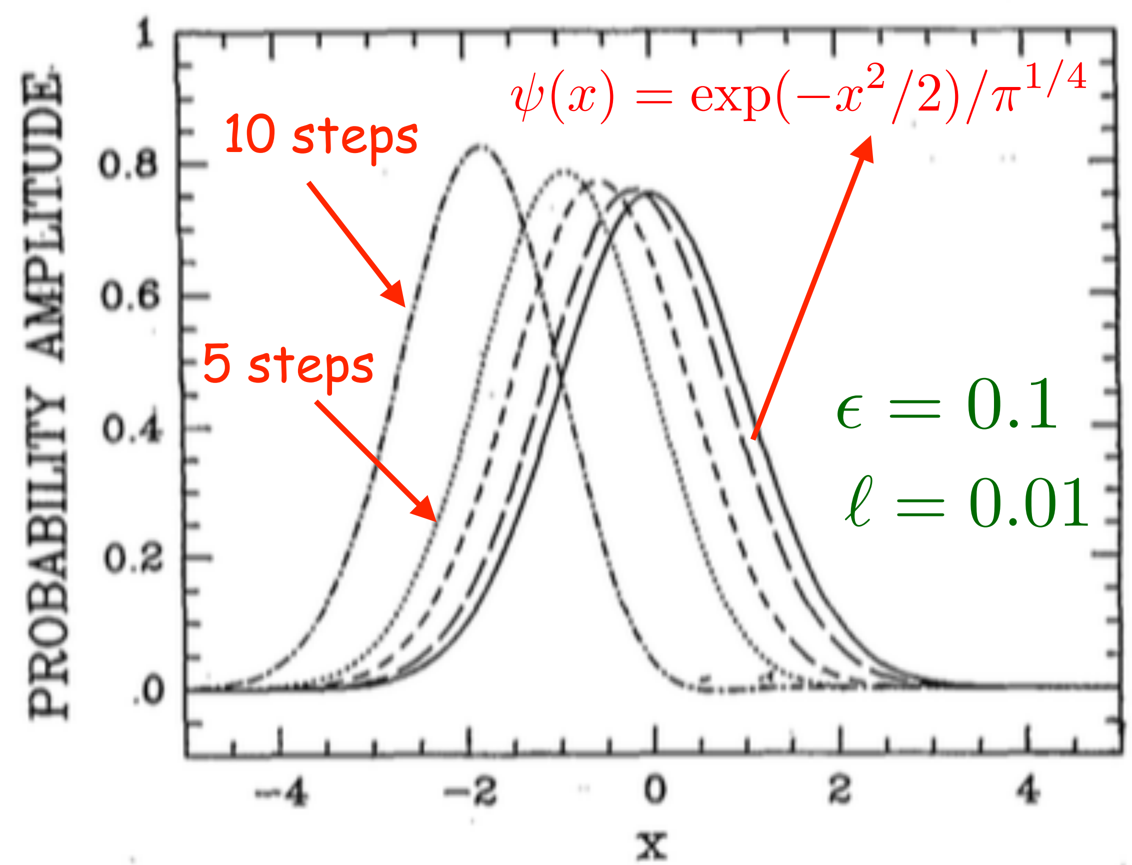
Final state $|\Psi\rangle = c_\uparrow|\uparrow\rangle|\psi(x_0-\ell)\rangle + c_\downarrow|\downarrow\rangle|\psi(x_0+\ell)\rangle)$

Final state in configuration space conditioned to measurement of state

$$|\theta\rangle = \sin(\theta/2)|\uparrow\rangle - \cos(\theta/2)|\downarrow\rangle$$

with $\tan(\theta/2) = |c_\downarrow/c_\uparrow|(1-\epsilon)$

(For $\epsilon = 0$, $|\theta\rangle$ becomes orthogonal to initial spin state)



What about the precision?

Quantum Fisher information corresponding to g (averages in initial state):

$$\hat{U}(g) = \exp(-ig\hat{A}\otimes\hat{M}) \Rightarrow \mathcal{F}(g) = 4\left[\langle\hat{A}^2\rangle\langle\hat{M}^2\rangle - \langle\hat{A}\rangle^2\langle\hat{M}\rangle^2\right] \to \mathcal{F}(g) = 4\langle\hat{A}^2\rangle\langle\hat{M}^2\rangle$$

(Assume ${}_M\langle\phi_i|\hat{M}|\phi_i\rangle_M = 0$)

Corresponding Fisher information with post-selection procedure

POVM: $\{|\psi_f\rangle\langle\psi_f|\otimes\hat{E}_j, (\hat{1}_{\mathcal{A}} - |\psi_f\rangle\langle\psi_f|)\otimes\hat{1}_{\mathcal{M}}\}, \quad j = 1, 2, \ldots n$

$$F_{ps}(g) = \underbrace{p_f(g)\sum_{j=1}^{n}\frac{1}{P_j(g)}\left[\frac{dP_j(g)}{dg}\right]^2}_{\text{Fisher information corresponding to measurements on the meter after post-selection, degraded by loss of statistical data}} + \underbrace{\frac{1}{p_f(g)[1-p_f(g)]}\left[\frac{dp_f(g)}{dg}\right]^2}_{\text{Information on g encoded in } p_f(g)}$$

$$\boxed{P_j(g) = \langle\phi_f(g)|\hat{E}_j|\phi_f(g)\rangle}$$

$$p_f(g) = |\langle\psi_f|\hat{U}(g)|\Psi_i\rangle|^2$$

Fisher information corresponding to measurements on the meter after post-selection, degraded by loss of statistical data

Information on g encoded in $p_f(g)$

$|\psi_f\rangle \to$ Post-selected state of A

$\hat{E}_j \to$ Generalized measurements on M

PHYSICAL REVIEW A **91**, 062107 (2015)

Weak-value amplification as an optimal metrological protocol

G. Bié Alves, B. M. Escher, R. L. de Matos Filho, N. Zagury, and L. Davidovich

Instituto de Física, Universidade Federal do Rio de Janeiro, P.O. Box 68528, Rio de Janeiro, RJ 21941-972, Brazil

What about the precision?

$$F_{ps}(g) = \underbrace{p_f(g)\sum_{j=1}^{n}\frac{1}{P_j(g)}\left[\frac{dP_j(g)}{dg}\right]^2}_{F_M(g)} + \underbrace{\frac{1}{p_f(g)[1-p_f(g)]}\left[\frac{dp_f(g)}{dg}\right]^2}_{F_{p_f}(g)}$$

$$\boxed{P_j(g) = \langle\phi_f(g)|\hat{E}_j|\phi_f(g)\rangle}$$

Fisher information corresponding to measurements on the meter after post-selection, degraded by loss of statistical data

Information on g encoded in $p_f(g)$

$$p_f(g) = |\langle\psi_f|\hat{U}(g)|\Psi_i\rangle|^2$$

Optimal post-selection: $|\psi_f^{\text{opt}}\rangle = \dfrac{\hat{A}|\psi_i\rangle}{\langle\hat{A}^2\rangle^{1/2}} \Rightarrow F_{ps}(g) \rightarrow$ Quantum Fisher information $\mathcal{F}(g) = 4\langle\hat{A}^2\rangle\langle\hat{M}^2\rangle$ for small g

$|gA_w|\Delta M \ll 1 \implies F_M(g) \rightarrow \mathcal{F}(g)$ Region of validity of weak-value theory

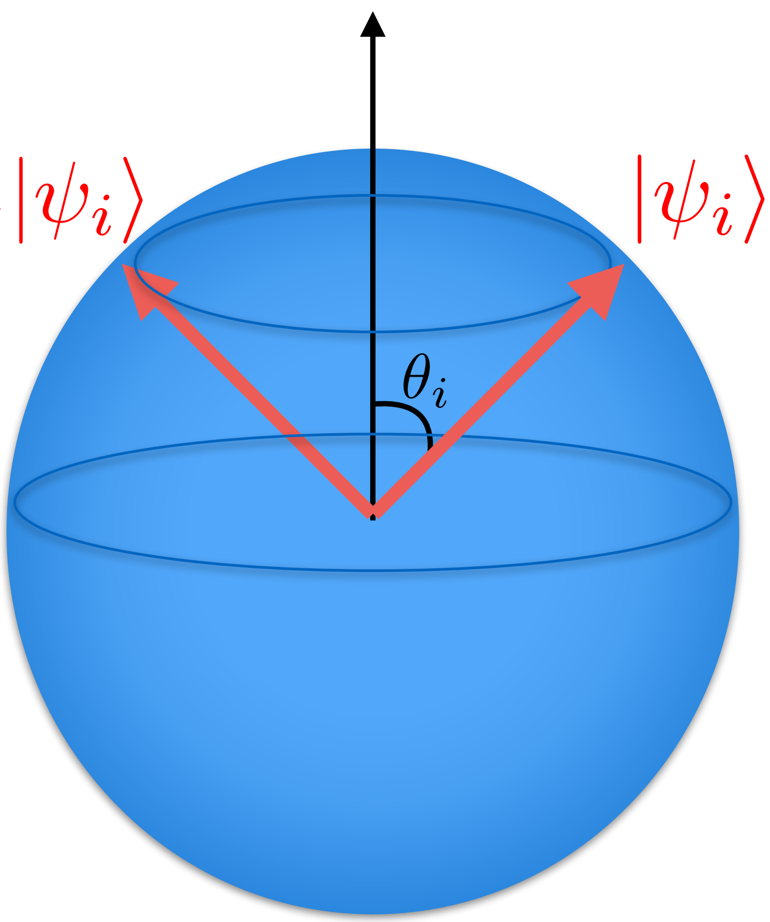
$|gA_w|\Delta M \gg 1 \implies F_{p_f}(g) \rightarrow \mathcal{F}(g)$ Region $|\langle\psi_f|\psi_i\rangle| \ll 1$

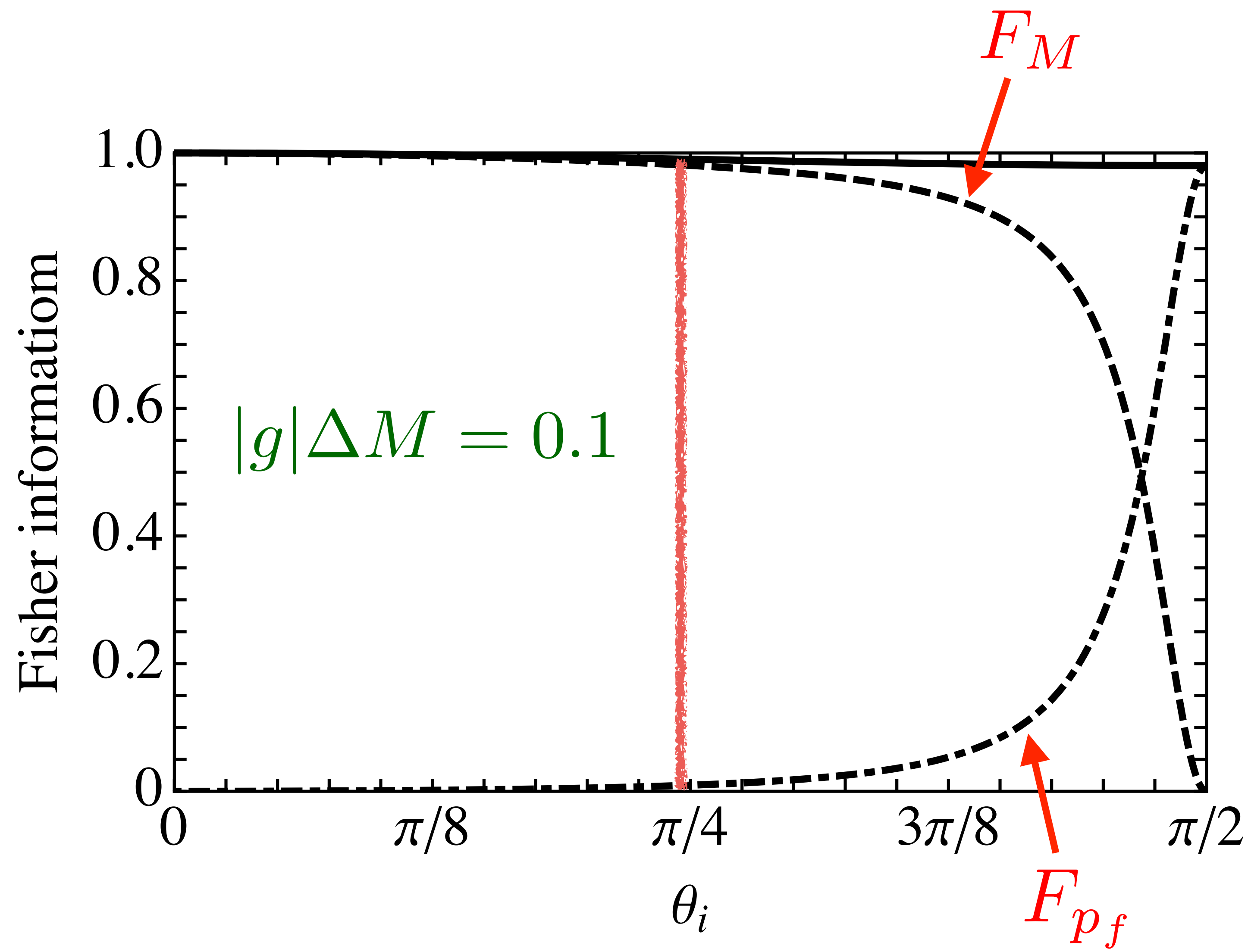
Weak value: $A_w = \dfrac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \dfrac{\langle\psi_i|\hat{A}^2|\psi_i\rangle}{\langle\psi_i|\hat{A}|\psi_i\rangle} \geq \langle\psi_i|\hat{A}|\psi_i\rangle$

Example: spin measurement

$\hat{A} = \hat{\sigma}_z \quad \hat{A}^2 = \hat{1} \quad \hat{U}(g) = \exp\left(-ig\hat{\sigma}_z \hat{M}\right) \quad |\psi_f^{\rm opt}\rangle = \frac{\hat{A}|\psi_i\rangle}{\langle \hat{A}^2\rangle^{1/2}} = \hat{\sigma}_z|\psi_i\rangle$

$|\psi_f\rangle = \hat{\sigma}_z|\psi_i\rangle$

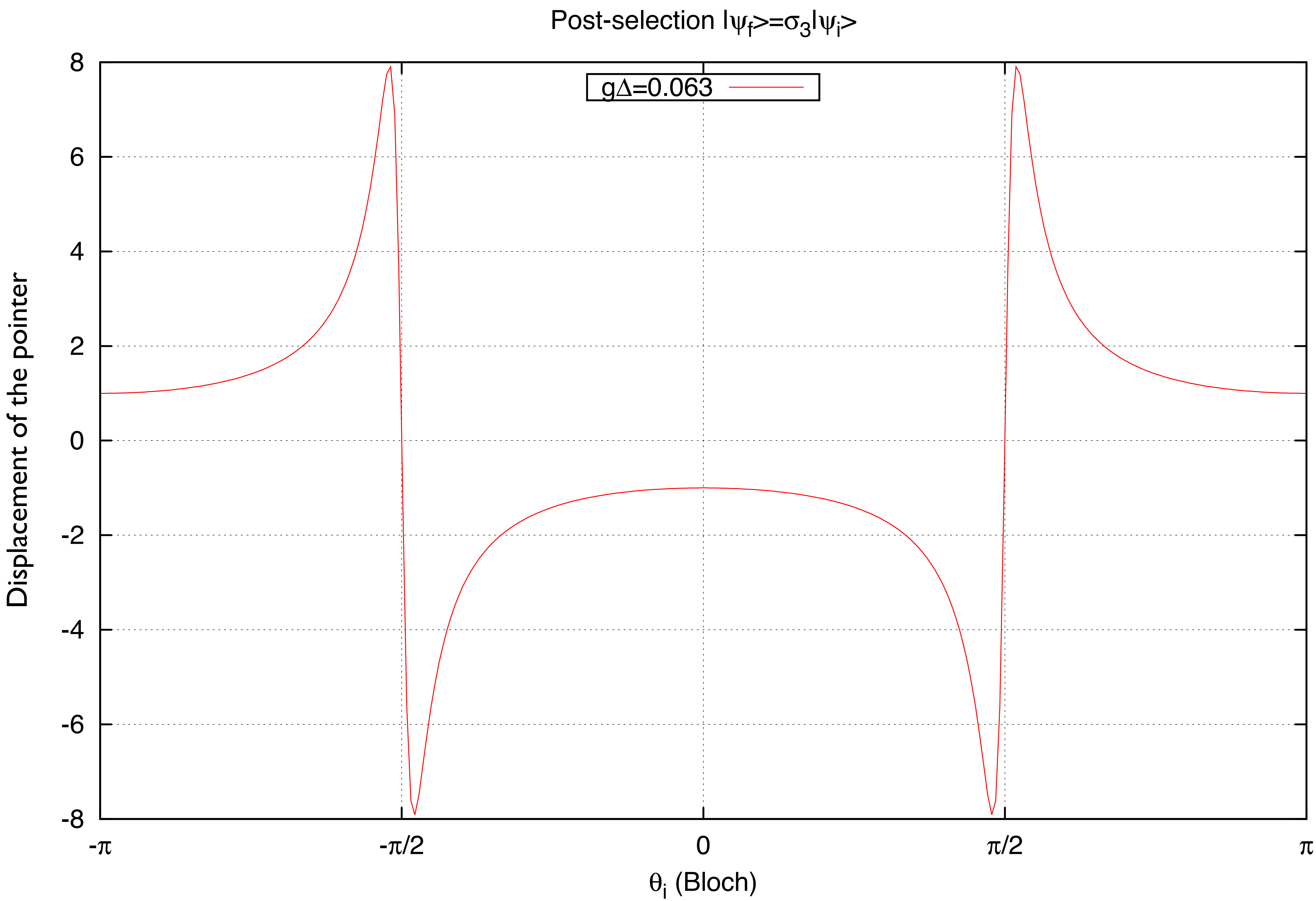




$$A_w = \frac{1}{\underbrace{\langle \psi_f|\psi_i\rangle}_{\delta}} = \frac{1}{\langle \psi_i|\hat{\sigma}_z|\psi_i\rangle}$$

Transition region: $|gA_w|\Delta M \approx 1$

Example: spin measurement



Weak Value Amplification is Suboptimal for Estimation and Detection

Christopher Ferrie and Joshua Combes
Center for Quantum Information and Control, University of New Mexico, Albuquerque, New Mexico 87131-0001, USA
(Received 25 July 2013; revised manuscript received 21 November 2013; published 31 January 2014)

We show by using statistically rigorous arguments that the technique of weak value amplification does not perform better than standard statistical techniques for the tasks of single parameter estimation and signal detection. Specifically, we prove that postselection, a necessary ingredient for weak value amplification, decreases estimation accuracy and, moreover, arranging for anomalously large weak values is a suboptimal strategy. In doing so, we explicitly provide the optimal estimator, which in turn allows us to identify the optimal experimental arrangement to be the one in which all outcomes have equal weak values (all as small as possible) and the initial state of the meter is the maximal eigenvalue of the square of the system observable. Finally, we give precise quantitative conditions for when weak measurement (measurements without postselection or anomalously large weak values) can mitigate the effect of uncharacterized technical noise in estimation.

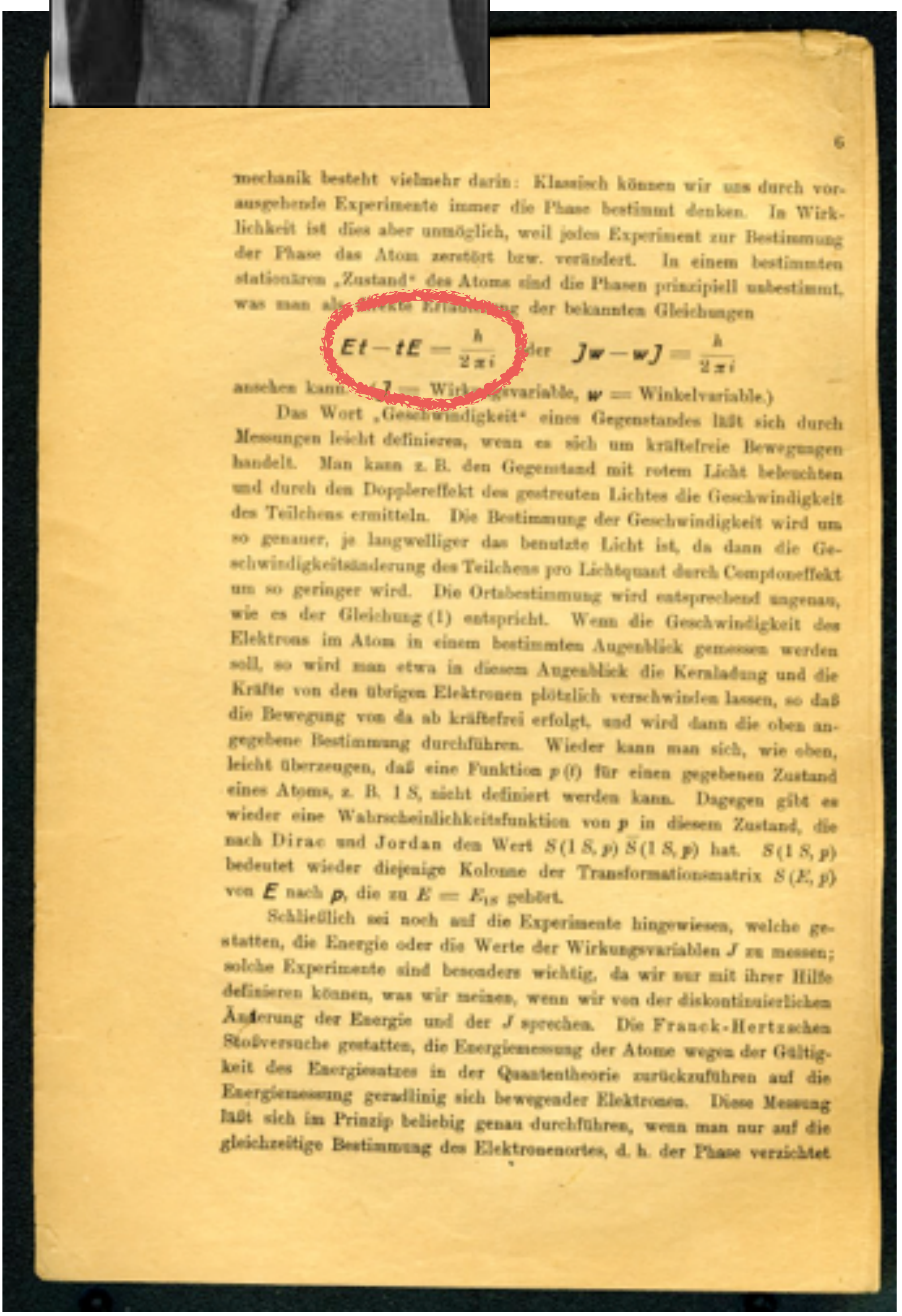
Technical Advantages for Weak-Value Amplification: When Less Is More

Andrew N. Jordan,[1,2] Julián Martínez-Rincón,[1] and John C. Howell[1]
[1]*Department of Physics and Astronomy and The Center for Coherence and Quantum Optics, University of Rochester, Rochester, New York 14627, USA*
[2]*Institute for Quantum Studies, Chapman University, 1 University Drive, Orange, California 92866, USA*
(Received 19 September 2013; published 6 March 2014)

The technical merits of weak-value-amplification techniques are analyzed. We consider models of several different types of technical noise in an optical context and show that weak-value-amplification techniques (which only use a small fraction of the photons) compare favorably with standard techniques (which use all of them). Using the Fisher-information metric, we demonstrate that weak-value techniques can put all of the Fisher information about the detected parameter into a small portion of the events and show how this fact alone gives technical advantages. We go on to consider a time-correlated noise model and find that a Fisher-information analysis indicates that the standard method can have much larger information about the detected parameter than the postselected technique. However, the estimator needed to gather the information is technically difficult to implement, showing that the inefficient (but practical) signal-to-noise estimation of the parameter is usually superior. We also describe other technical advantages unique to imaginary weak-value-amplification techniques, focusing on beam-deflection measurements. In this case, we discuss combined noise types (such as detector transverse jitter, angular beam jitter before the interferometer, and turbulence) for which the interferometric weak-value technique gives higher Fisher information over conventional methods. We go on to calculate the Fisher information of the recently proposed photon-recycling scheme for beam-deflection measurements and show it further boosts the Fisher information by the inverse postselection probability relative to the standard measurement case.

Energy-time uncertainty

$$\Delta E \Delta T \geq \hbar$$



6

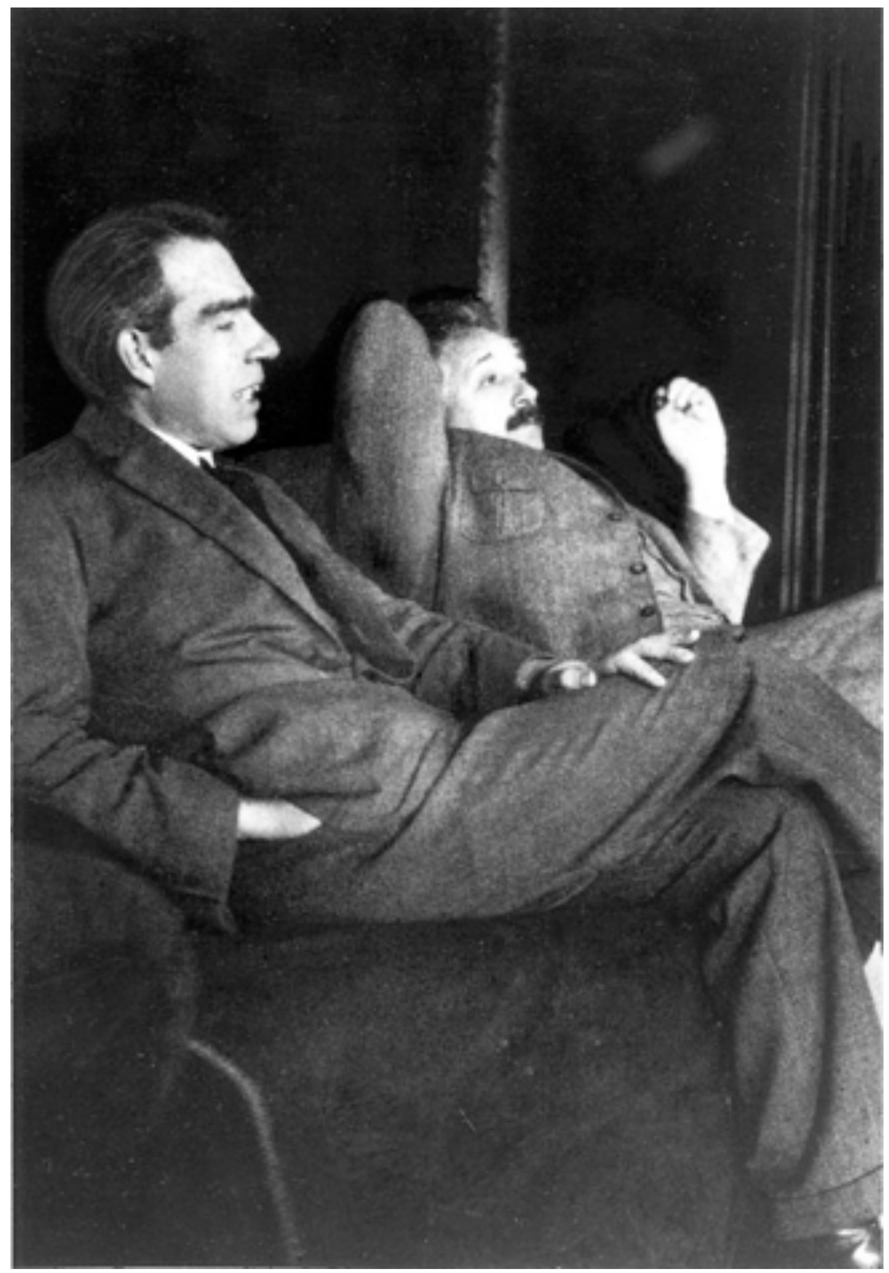
mechanik besteht vielmehr darin: Klassisch können wir uns durch vorausgehende Experimente immer die Phase bestimmt denken. In Wirklichkeit ist dies aber unmöglich, weil jedes Experiment zur Bestimmung der Phase das Atom zerstört bzw. verändert. In einem bestimmten stationären „Zustand" des Atoms sind die Phasen prinzipiell unbestimmt, was man als direkte Erläuterung der bekannten Gleichungen

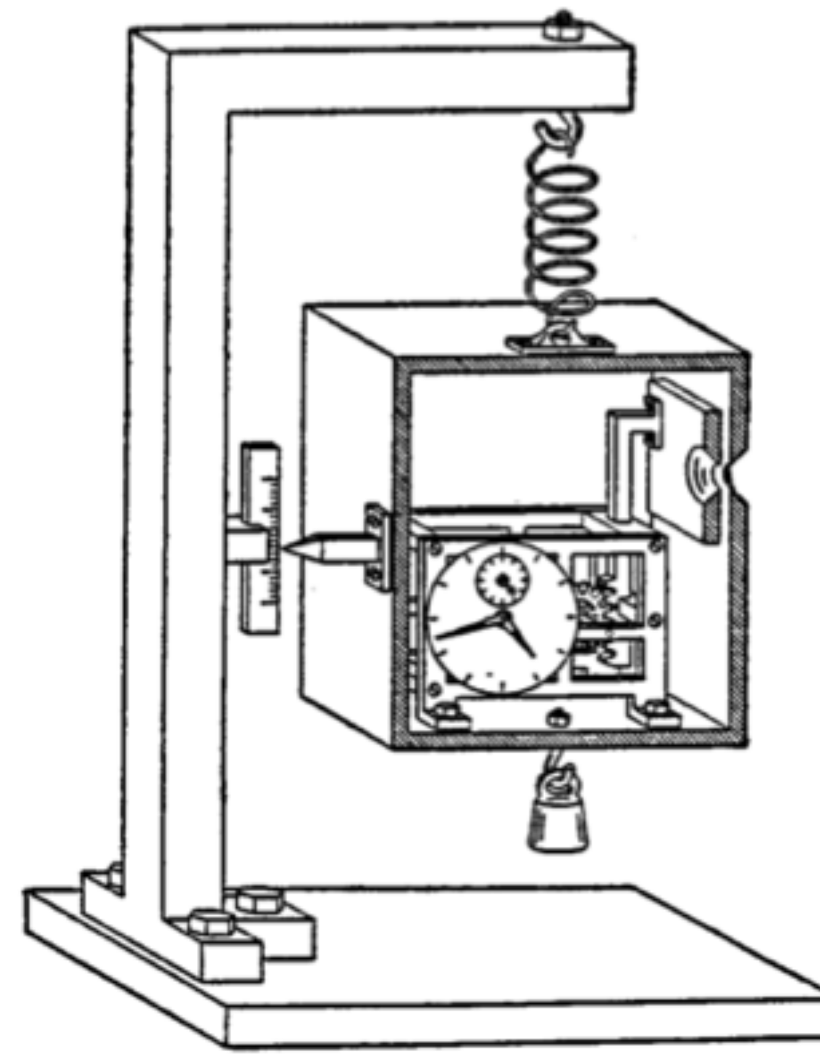
$$Et - tE = \frac{h}{2\pi i} \quad \text{oder} \quad Jw - wJ = \frac{h}{2\pi i}$$

ansehen kann (J = Wirkungsvariable, w = Winkelvariable.)

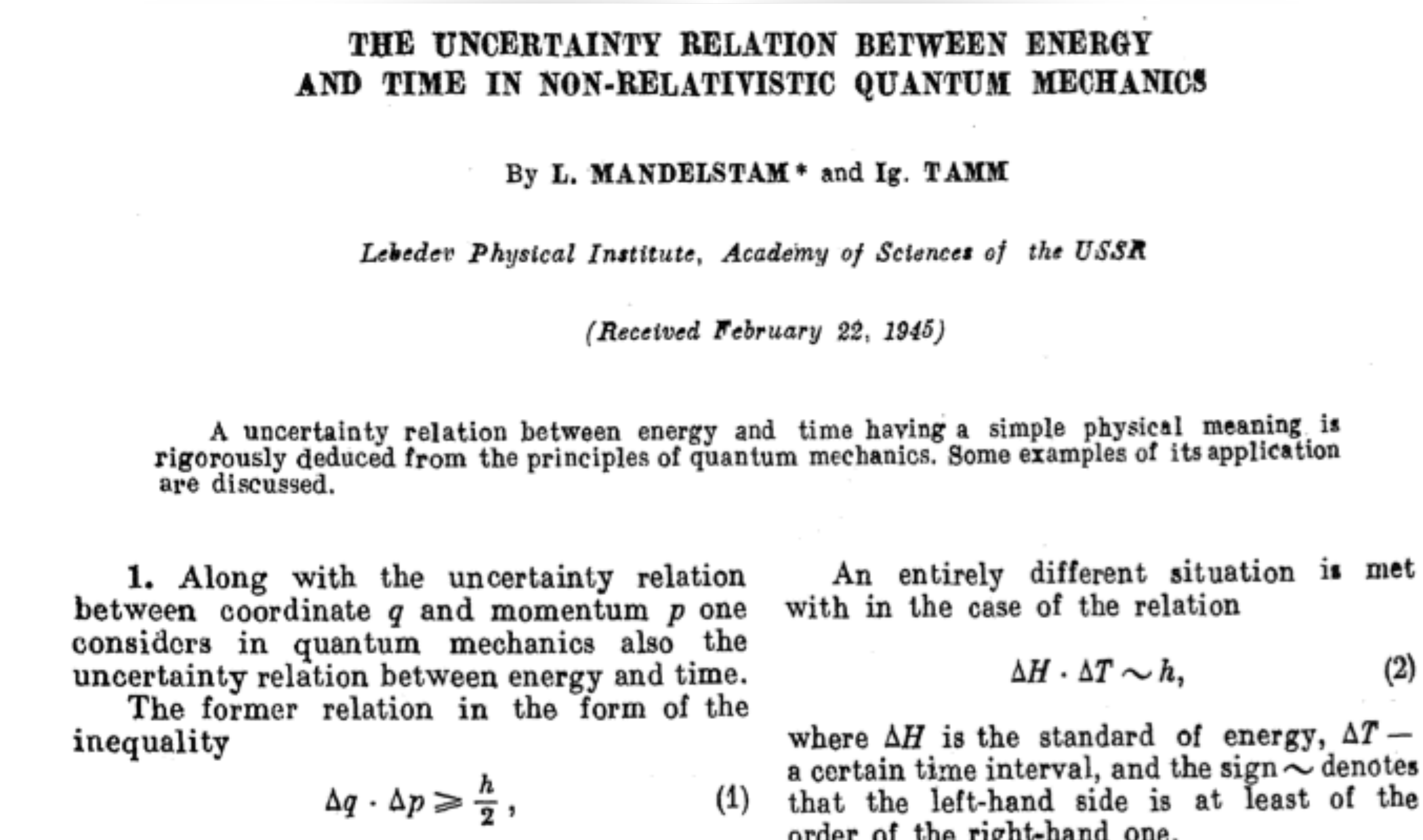
Das Wort „Geschwindigkeit" eines Gegenstandes läßt sich durch Messungen leicht definieren, wenn es sich um kräftefreie Bewegungen handelt. Man kann z. B. den Gegenstand mit rotem Licht beleuchten und durch den Dopplereffekt des gestreuten Lichtes die Geschwindigkeit des Teilchens ermitteln. Die Bestimmung der Geschwindigkeit wird um so genauer, je langwelliger das benutzte Licht ist, da dann die Geschwindigkeitsänderung des Teilchens pro Lichtquant durch Comptoneffekt um so geringer wird. Die Ortsbestimmung wird entsprechend ungenau, wie es der Gleichung (1) entspricht. Wenn die Geschwindigkeit des Elektrons im Atom in einem bestimmten Augenblick gemessen werden soll, so wird man etwa in diesem Augenblick die Kernladung und die Kräfte von den übrigen Elektronen plötzlich verschwinden lassen, so daß die Bewegung von da ab kräftefrei erfolgt, und wird dann die oben angegebene Bestimmung durchführen. Wieder kann man sich, wie oben, leicht überzeugen, daß eine Funktion $p(t)$ für einen gegebenen Zustand eines Atoms, z. B. $1S$, nicht definiert werden kann. Dagegen gibt es wieder eine Wahrscheinlichkeitsfunktion von p in diesem Zustand, die nach Dirac und Jordan den Wert $S(1S, p)\,\bar{S}(1S, p)$ hat. $S(1S, p)$ bedeutet wieder diejenige Kolonne der Transformationsmatrix $S(E, p)$ von E nach p, die zu $E = E_{1S}$ gehört.

Schließlich sei noch auf die Experimente hingewiesen, welche gestatten, die Energie oder die Werte der Wirkungsvariablen J zu messen; solche Experimente sind besonders wichtig, da wir nur mit ihrer Hilfe definieren können, was wir meinen, wenn wir von der diskontinuierlichen Änderung der Energie und der J sprechen. Die Franck-Hertzschen Stoßversuche gestatten, die Energiemessung der Atome wegen der Gültigkeit des Energiesatzes in der Quantentheorie zurückzuführen auf die Energiemessung geradlinig sich bewegender Elektronen. Diese Messung läßt sich im Prinzip beliebig genau durchführen, wenn man nur auf die gleichzeitige Bestimmung des Elektronenortes, d. h. der Phase verzichtet





Energy-time uncertainty



THE UNCERTAINTY RELATION BETWEEN ENERGY AND TIME IN NON-RELATIVISTIC QUANTUM MECHANICS

By L. MANDELSTAM * and Ig. TAMM

Lebedev Physical Institute, Academy of Sciences of the USSR

(Received February 22, 1945)

A uncertainty relation between energy and time having a simple physical meaning is rigorously deduced from the principles of quantum mechanics. Some examples of its application are discussed.

1. Along with the uncertainty relation between coordinate q and momentum p one considers in quantum mechanics also the uncertainty relation between energy and time.

The former relation in the form of the inequality

$$\Delta q \cdot \Delta p \geqslant \frac{h}{2}, \qquad (1)$$

An entirely different situation is met with in the case of the relation

$$\Delta H \cdot \Delta T \sim h, \qquad (2)$$

where ΔH is the standard of energy, ΔT — a certain time interval, and the sign $\sim$ denotes that the left-hand side is at least of the order of the right-hand one.

Leonid Mandelstam

Igor Tamm

Energy-time uncertainty

VOLUME 65, NUMBER 14 PHYSICAL REVIEW LETTERS 1 OCTOBER 1990

Geometry of Quantum Evolution

J. Anandan[(a)]
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom

Y. Aharonov[(b)]
Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208

Geometric derivation, generalization to time-dependent Hamiltonians. Inequality derived from the condition that actual path followed by the states should be larger than geodesic connecting the two states.

Generalization to non-unitary processes? Life-time for decay processes? Hamiltonian should not show up!

Motivation

1. Foundations of quantum mechanics: How to interpret this relation? (Heisenberg, Einstein, Bohr, Mandelstam and Tamm, Landau and Peierls, Fock and Krylov, Aharonov and Bohm, Bhattacharyya)
2. Computation times: e.g., time taken to flip a spin — Quantum speed limit
3. Quantum-classical transition: Decoherence time
4. Control of the dynamics of a quantum system: find the fastest evolution given initial and final states and some restriction on the resources (e.g. the energy) or the general structure of the Hamiltonian.
5. Relation with quantum metrology

Quantum speed limit for physical processes

M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, PRL **110**, 050402 (2013)

$$\arccos\sqrt{\Phi_B[\hat\rho(0),\hat\rho(\tau)]} \le \int_0^\tau \sqrt{\mathcal{F}_Q(t)}/2dt$$

$\Rightarrow$ Lower bound for time needed to reach fidelity $\Phi_B[\hat\rho(0),\hat\rho(\tau)]$ between initial and final states

Bures length of geodesic (Distance between two states)

Bures length of actual path followed by state of the system

Inequality is saturated if dynamical process joins $\rho(0)$ and $\rho(\tau)$ along a geodesic

Special case: Unitary evolution, time-independent Hamiltonian, orthogonal states

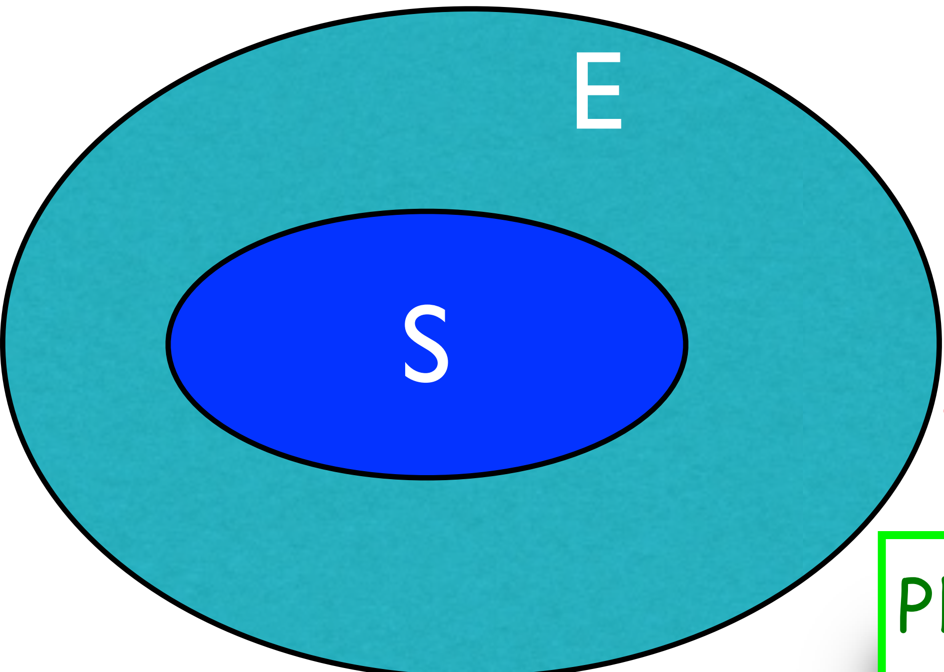
Anandan-Aharonov

$$\Phi_B[\hat\rho(0),\hat\rho(\tau)] = 0, \quad \mathcal{F}_Q(t) = 4\langle(\Delta H)^2\rangle/\hbar^2 \Rightarrow \boxed{\tau\sqrt{\langle(\Delta H)^2\rangle} \ge h/4}$$

Parameter estimation in open systems: Extended space approach

B. M. Escher, R. L. Matos Filho, and L. D., **Nature Physics** 7, 406 (2011); **Braz. J. Phys.** 41, 229 (2011)

Given initial state and non-unitary evolution, define in S+E



$$|\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x)|\psi\rangle_S|0\rangle_E$$ (Purification)

Then

$$\mathcal{F}_Q \equiv \max_{\hat{E}_j^{(S)}\otimes\hat{1}} F\left(\hat{E}_j^{(S)}\otimes\hat{1}\right) \le \max_{\hat{E}_j^{(S,E)}} F\left(\hat{E}_j^{(S,E)}\right) \equiv \mathcal{C}_Q$$

Physical meaning of this bound: information obtained about parameter when S+E is monitored

Least upper bound: Minimization over all unitary evolutions in S+E - difficult problem

Bound is attainable - there is always a purification such that $\mathcal{C}_Q = \mathcal{F}_Q$

Then, monitoring S+E yields same information as monitoring S

Quantum speed limit for open systems: Purification procedure

$$\mathcal{D} := \arccos\sqrt{\Phi_B\left[\hat{\rho}(0), \hat{\rho}(\tau)\right]} \leq \int_0^\tau \sqrt{\mathcal{F}_Q(t)/4}\, dt$$

$\Downarrow$

Problem: No analytical expression for $\mathscr{F}_Q$ $\Longrightarrow$ Purification!

$$\mathcal{D} \leq \int_0^\tau \sqrt{\mathcal{C}_Q(t)/4}\, dt = \int_0^\tau \sqrt{\langle \Delta \hat{\mathcal{H}}^2_{S,E}(t)\rangle}/\hbar\, dt.$$

$$\hat{\mathcal{H}}_{S,E}(t) := \frac{\hbar}{i}\frac{d\hat{U}^\dagger_{S,E}(t)}{dt}\hat{U}_{S,E}(t)$$

$\hat{U}_{S,E}(t)$: Evolution of purified state corresponding to $\hat{\rho}_S$

Quantum speed limit for physical processes: amplitude damping channel

$$|0\rangle|0\rangle_E \rightarrow |0\rangle|0\rangle_E \,,$$

$$|1\rangle|0\rangle_E \rightarrow \sqrt{P(t)}|1\rangle|0\rangle_E + \sqrt{1-P(t)}|0\rangle|1\rangle_E \quad P(t) = \exp(-\gamma t)$$

$$\mathcal{D} \leq \sqrt{\langle\hat{\sigma}_+\hat{\sigma}_-\rangle}\,\arccos[\exp(-\gamma\tau/2)] \Rightarrow \boxed{\gamma\tau \geq 2\ln\sec(\mathcal{D}/\sqrt{\langle\hat{\sigma}_+\hat{\sigma}_-\rangle})}$$

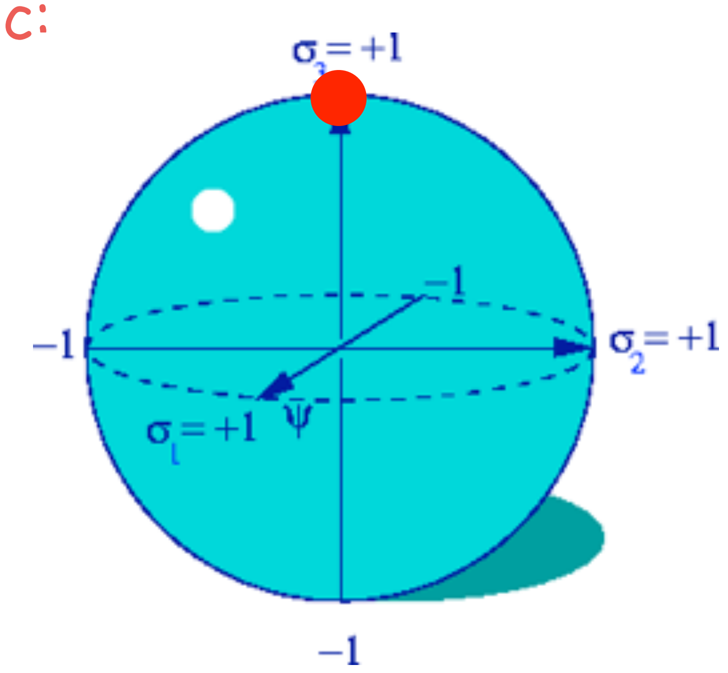
Initial population of excited state

Bound is saturated if $\langle\hat{\sigma}_+\hat{\sigma}_-\rangle = 0 \;\text{ or }\; 1$

Interpretation:

If initial state is the excited state, then evolution is along a geodesic:

$$\langle\hat{\sigma}_+\hat{\sigma}_-\rangle = 1 \Rightarrow |1\rangle\langle 1| \rightarrow P(t)|1\rangle\langle 1| + [1-P(t)]|0\rangle\langle 0|$$



Also good for dephasing channel

Related work

ARTICLES
PUBLISHED ONLINE: 27 MARCH 2011 | DOI: 10.1038/NPHYS1958
nature physics

General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology

B. M. Escher*, R. L. de Matos Filho and L. Davidovich

→ Phase estimation in lossy optical interferometers

PRL 109, 190404 (2012) PHYSICAL REVIEW LETTERS week ending 9 NOVEMBER 2012

Quantum Metrological Limits via a Variational Approach

B. M. Escher,* L. Davidovich, N. Zagury, and R. L. de Matos Filho

→ Phase diffusion in optical interferometers

PHYSICAL REVIEW A **88**, 042112 (2013)

Quantum limit for the measurement of a classical force coupled to a noisy quantum-mechanical oscillator

C. L. Latune, B. M. Escher, R. L. de Matos Filho, and L. Davidovich*

→ Estimation of weak forces

Collaborators: quantum metrology


Gabriel Bié


Marcio Taddei


Camille Latune


Bruno Escher


Alvaro Pimentel


Nicim Zagury


Ruynet Matos Filho


Stephen Walborn


Malena Hor-Meyll