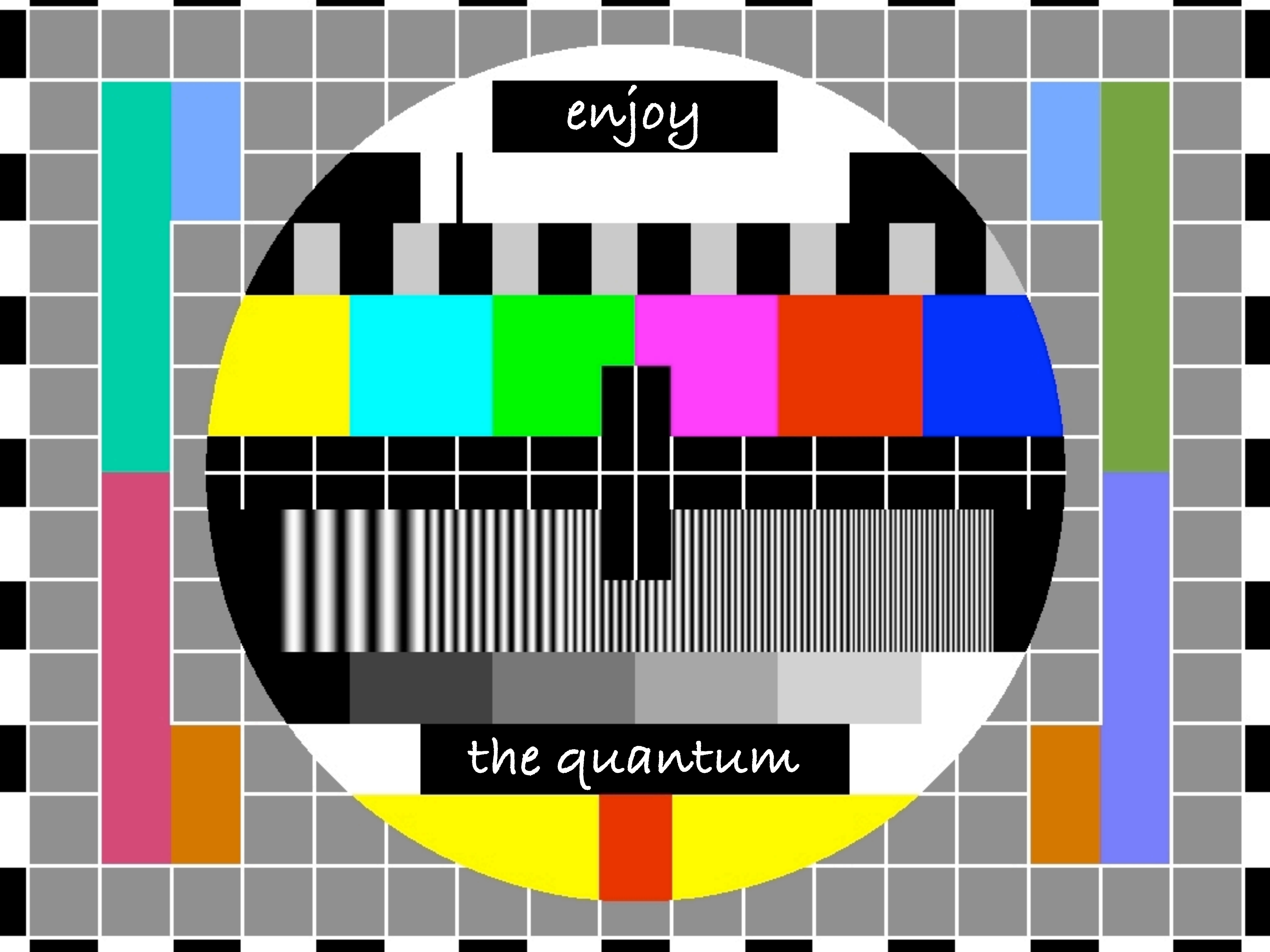
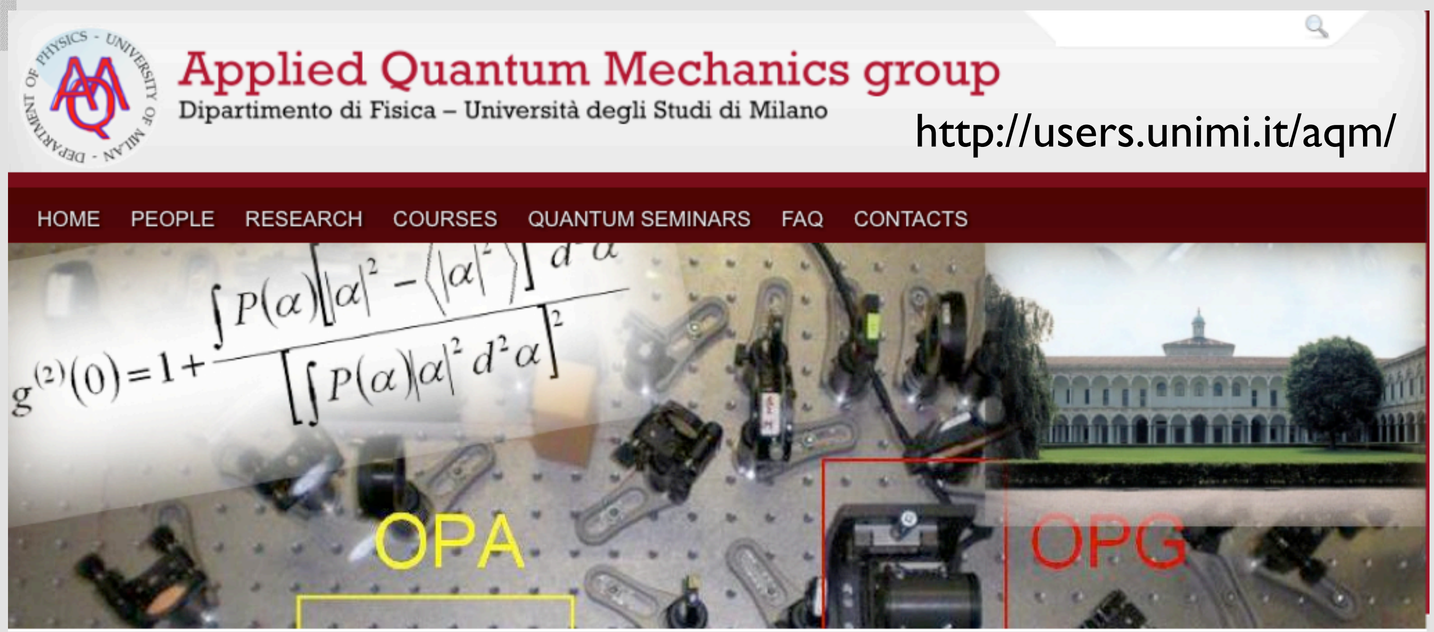

enjoy
the quantum

Quantum estimation for quantum technology

Matteo G A Paris



GDR - IQFA Grenoble, 29 November 2012

Quantum technology

- Quantum information
  (communication and computing)

- Quantum metrology
  (calibration, interferometry, nanopositioning)

- Quantum imaging
  (ghost imaging and diffraction, quantum litography)

Quantum characterization for quantum technology

- It is highly desirable to have theoretical and experimental tools for the precise characterization of signal and devices at the quantum level

Quantum estimation

- The "resources" involved in quantum-enahnced metrology/technology are entanglement, nonlocality, entropy, interferometric phase-shift, etc..

Quantum estimation

- The "resources" involved in quantum-enahnced metrology/technology are entanglement, nonlocality, entropy, interferometric phase-shift, etc..

  In general they are not observable quantities in strict sense (do not correspond to a selfadjoint operator)

  - No correspondence principle
  - No uncertainty relations

Quantum estimation

- The "resources" involved in quantum-enahnced metrology/technology are entanglement, nonlocality, entropy, interferometric phase-shift, etc..

  In general they are not observable quantities in strict sense (do not correspond to a selfadjoint operator)

Quantum estimation theory

Quantum estimation

ϱ_λ

$\{\Pi_x\}_{x\in\mathcal{X}}$

$\chi = (x_1, x_2, \dots)$

- Optimal measurements
- Ultimate bounds to precision

Quantum estimation

$$\left(\varrho_0 \; \boxed{\Gamma_\lambda}\;\right) \qquad \varrho_\lambda \rightsquigarrow \quad \{\Pi_x\}_{x\in\mathcal{X}}$$

$$\boldsymbol{\chi} = (x_1, x_2, \dots)$$

- Optimal measurements
- Ultimate bounds to precision

Measurement and estimation

- ~~direct measurements~~
- indirect measurements

influence on a different quantity

$$\boldsymbol{x} = (x_1, x_2, \ldots)$$

$$\boldsymbol{x} \mapsto \hat{\lambda} = f(\boldsymbol{x})$$

- choice of the measurement

- choice of the estimator

Cramer - Rao bound (unbiased estimators)

- variance of unbiased estimators

$$\mathrm{Var}_\lambda[\widehat{\lambda}] \geq \frac{1}{MF(\lambda)}$$

- M -> number of measurements
- F -> Fisher Information

$$F(\lambda) = \int dx\, p(x|\lambda) \Big[\partial_\lambda \log p(x|\lambda)\Big]^2$$

- Optimal measurement -> maximum Fisher
- Optimal estimator -> saturation of CR inequality

(classical) Bayesian estimators (1)

- Bayes theorem $p(x|\lambda)p(\lambda) = p(\lambda|x)p(x)$

- M indipendent events: a posteriori distribution

$$p(\lambda|\{x\}) = \frac{1}{N}\prod_{k=1}^{M} p(x_k|\lambda) \qquad N = \int d\lambda \prod_{k=1}^{M} p(x_k|\lambda)$$

- Bayesian estimator: $\lambda_B = \int d\lambda\, \lambda\, p(\lambda|\{x\})$

mean of the a posteriori distribution

(classical) Bayesian estimators (2)

- Laplace - Bernstein - von Mises theorem

$$p(\lambda|\{x\}) \xrightarrow{M \gg 1} G(\lambda^*, \sigma^2)$$

- Bayes estimator is asymptotically efficient

$$\sigma^2 = \frac{1}{MF(\lambda^*)}$$

MaxLik estimation

- Probability distribution $p(x|\lambda)$
- Random sample $x_1, x_2, ..., x_M$
- Joint probability of the sample

$$\mathcal{L}(x_1, x_2, ...x_M|\lambda) = \Pi_{k=1}^{M} p(x_k|\lambda)$$

Maxlik estimation ➞ take the value of the parameters which maximize the likelihood of the observed data

Optimal estimation scheme (classical)

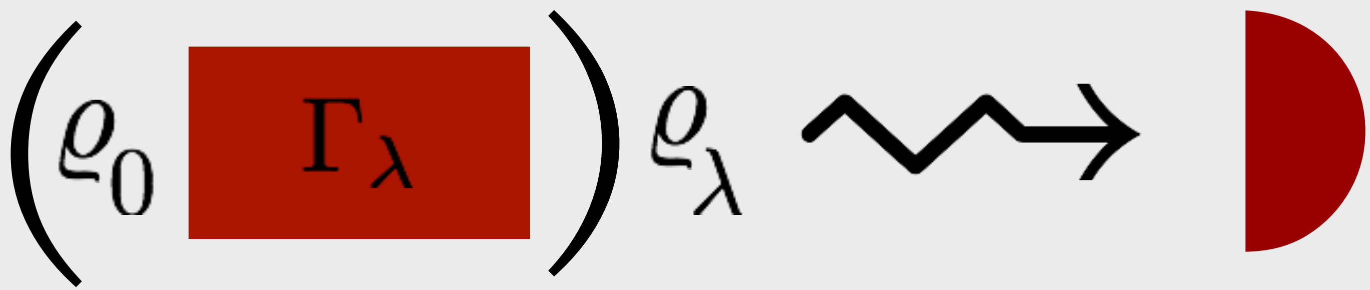
$$\boldsymbol{\chi} = (x_1, x_2, \dots)$$

$$\boldsymbol{\chi} \mapsto \widehat{\lambda} = f(\boldsymbol{\chi})$$

- Optimal measurement -> maximum Fisher (no recipes on how to find it)

- Optimal estimator -> saturation of CR inequality (e.g. Bayesian or MaxLik asymptotically)

Let's go quantum (1)



$$\{\Pi_x\}_{x\in\mathcal{X}}$$

$$\mathcal{X} = (x_1, x_2, \dots)$$

- probability density $p(x|\lambda) = \mathrm{Tr}\left[\varrho_\lambda\, \Pi_x\right]$

- symm. log. derivative (SLD) $$\frac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \frac{\partial \varrho_\lambda}{\partial \lambda}$$

  selfadjoint, zero mean $\mathrm{Tr}\left[\varrho_\lambda\, L_\lambda\right] = 0$

- Fisher information $$F(\lambda) = \int dx\, \frac{\mathrm{Re}\left(\mathrm{Tr}\left[\varrho_\lambda \Pi_x L_\lambda\right]\right)^2}{\mathrm{Tr}\left[\varrho_\lambda \Pi_x\right]}$$

Let's go quantum (2)

$$
\begin{aligned}
F(\lambda) &\leq \int dx \left| \frac{\mathrm{Tr}\left[\varrho_\lambda \Pi_x L_\lambda\right]}{\sqrt{\mathrm{Tr}[\varrho_\lambda \Pi_x]}} \right|^2 \\
&= \int dx \left| \mathrm{Tr}\left[\frac{\sqrt{\varrho_\lambda}\sqrt{\Pi_x}}{\sqrt{\mathrm{Tr}\left[\varrho_\lambda \Pi_x\right]}} \sqrt{\Pi_x} L_\lambda \sqrt{\varrho_\lambda} \right] \right|^2 \\
&\leq \int dx \, \mathrm{Tr}\left[\Pi_x L_\lambda \varrho_\lambda L_\lambda\right] \\
&= \mathrm{Tr}[L_\lambda \varrho_\lambda L_\lambda] = \mathrm{Tr}[\varrho_\lambda L_\lambda^2]
\end{aligned}
$$

Helstrom 1976

Braunstein & Caves 1994

- Fisher vs Quantum Fisher

$$F(\lambda) \leq H(\lambda) \equiv \mathrm{Tr}[\varrho_\lambda L_\lambda^2] = \mathrm{Tr}[\partial_\lambda \varrho_\lambda L_\lambda]$$

Optimal quantum measurement (1)

- ultimate bound on precision $\mathrm{Var}(\lambda) \geq \frac{1}{M H(\lambda)}$

- optimal POVM $\frac{\sqrt{\Pi_x}\sqrt{\varrho_\lambda}}{\mathrm{Tr}\left[\varrho_\lambda \Pi_x\right]} = \frac{\sqrt{\Pi_x} L_\lambda \sqrt{\varrho_\lambda}}{\mathrm{Tr}[\varrho_\lambda \Pi_x L_\lambda]} \quad \forall\lambda$

  eigenstates of the SLD

- optimal quantum estimation: projective measurement over the eigenstates of SLD + classical postprocessing (Bayesian, ML)

Optimal quantum measurement (2)

1. The optimal measurement and the bound do depend on the value of the parameter

- feedback assisted measurements
- one-step adaptive procedure: rough estimate of the parameter on a small fraction of copies + measurement of SLD on the rest of the copies

2. We are dealing with repeated measurement

- one parameter -> separate measurements

(Gill and Massar 2000)

General formulas (basis indepedent)

ϱ_λ 

$$\frac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \frac{\partial \varrho_\lambda}{\partial \lambda}$$

Lyapunov equation

- Symmetric logarithmic derivative

$$L_\lambda = 2\int_0^\infty dt\, \exp\{-\varrho_\lambda t\}\, \partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}$$

- Quantum Fisher Information

$$H(\lambda) = 2\int_0^\infty dt\, \mathrm{Tr}\left[\partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}\, \partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}\right]$$

General formulas

- **Family of quantum states**

$$\varrho_\lambda = \sum_n \varrho_n |\psi_n\rangle\langle\psi_n|$$

- **Symmetric logarithmic derivative**

$$L_\lambda = \sum_p \frac{\partial_\lambda \varrho_p}{\varrho_p} |\psi_p\rangle\langle\psi_p| + 2 \sum_{n\neq m} \frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m} \langle\psi_m|\partial_\lambda\psi_n\rangle \, |\psi_m\rangle\langle\psi_n|$$

- **Quantum Fisher Information**

$$H(\lambda) = \sum_p \frac{(\partial_\lambda \varrho_p)^2}{\varrho_p} + 2 \sum_{n\neq m} \frac{(\varrho_n - \varrho_m)^2}{\varrho_n + \varrho_m} \left|\langle\psi_m|\partial_\lambda\psi_n\rangle\right|^2$$

unitary families of quantum states

$$\varrho_\lambda = U_\lambda\, \varrho_0\, U_\lambda^\dagger \qquad \varrho_0 = \sum \varrho_n |\varphi_n\rangle\langle\varphi_n|$$

$$U_\lambda = \exp\{-i\lambda G\} \qquad \partial_\lambda \varrho_\lambda = U_\lambda [G, \varrho_0] U_\lambda^\dagger$$

- covariance of SLD $\quad L_\lambda = U_\lambda\, L_0\, U_\lambda^\dagger$

$$L_0 = 2 \sum_{n,m} \frac{\langle\varphi_m|[G,\varrho_0]|\varphi_n\rangle}{\varrho_n + \varrho_m} |\varphi_n\rangle\langle\varphi_m|$$

- QFI is independent on the value of the parameter

$$H = \mathrm{Tr}\left[\varrho_0\, L_0^2\right] = 2 \sum_{n\neq m} \frac{(\varrho_n - \varrho_m)^2}{\varrho_n + \varrho_m} \langle\varphi_m|G|\varphi_n\rangle^2$$

parameter-based uncertainty relations

- pure states $H = 4\langle\psi_0|\Delta G^2|\psi_0\rangle$

$$\mathrm{Var}(\lambda)\langle\Delta G^2\rangle \geq \frac{1}{4M}$$

parameter-based uncertainty relations

- pure states $H = 4\langle\psi_0|\Delta G^2|\psi_0\rangle$

$$\mathrm{Var}(\lambda)\langle\Delta G^2\rangle \geq \frac{1}{4M}$$

- mixed states

$$H = 4\,\mathrm{Tr}\left[\Delta G^2 \varrho_0\right] + \boxed{4\sum_n \varrho_n \langle\varphi_n|\langle G\rangle^2 - 2GK^{(n)}G|\varphi_n\rangle}$$

$$K^{(n)} = \sum_m \frac{\varrho_m}{\varrho_n + \varrho_m}|\varphi_m\rangle\langle\varphi_m| \xrightarrow{\varrho_0 \to |\varphi_0\rangle\langle\varphi_0|} \frac{1}{2}|\varphi_0\rangle\langle\varphi_0|$$

$$\mathrm{Var}(\lambda)\langle\Delta G^2\rangle \geq \frac{1}{4M}\left[1 + \sum_n \varrho_n \langle\varphi_n|\langle G\rangle^2 - 2GK^{(n)}G|\varphi_n\rangle\right]^{-1}$$

estimability of a parameter

- signal-to-noise ratio (single measurement)

$$R_\lambda = \frac{\lambda^2}{\mathrm{Var}(\lambda)} \leq Q_\lambda \equiv \lambda^2 H(\lambda)$$

- relative error for a 3σ confidence interval (after M measurements)

$$\delta^2 = \frac{9\mathrm{Var}(\lambda)}{M\lambda^2} = \frac{9}{M}\frac{1}{Q_\lambda} = \frac{9}{M\lambda^2 H(\lambda)}$$

- # of meas to achieve a given relative error

$$M_\delta = \frac{9}{\delta^2}\frac{1}{Q_\lambda}$$

estimability of a parameter: the unitary case

$$\varrho_\lambda = U_\lambda \, \varrho_0 \, U_\lambda^\dagger$$

- QFI is independent on the value of the parameter
- (Any) estimation procedure cannot be efficient for small value of the parameter

$$Q_\lambda \propto \lambda^2 \qquad\qquad M_\delta \propto \frac{1}{\delta^2 \lambda^2}$$

A nonunitary example: estimation of loss

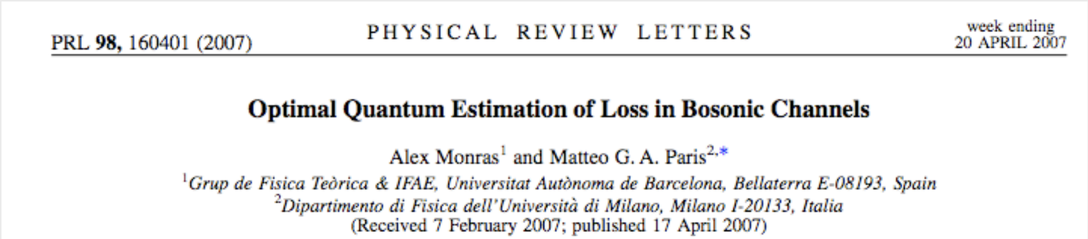
- Master equation $\frac{d\varrho_\phi}{d\phi} = \tan\phi \, L[a]\varrho_\phi \quad \exp\{-\gamma t\} = \cos^2\phi$

$$L[a]\varrho = 2a^\dagger \varrho a - a^\dagger a \varrho - \varrho a^\dagger a$$

- absorption
- propagation in a noisy channel (T=0)

$$\mathrm{Var}_\gamma[\hat{\gamma}] \to \frac{\gamma}{\bar{n}Mt} + \mathrm{O}(\gamma^2)$$

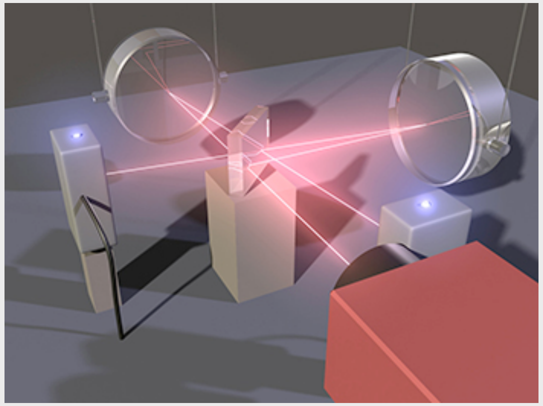
proportional to the loss parameter itself !

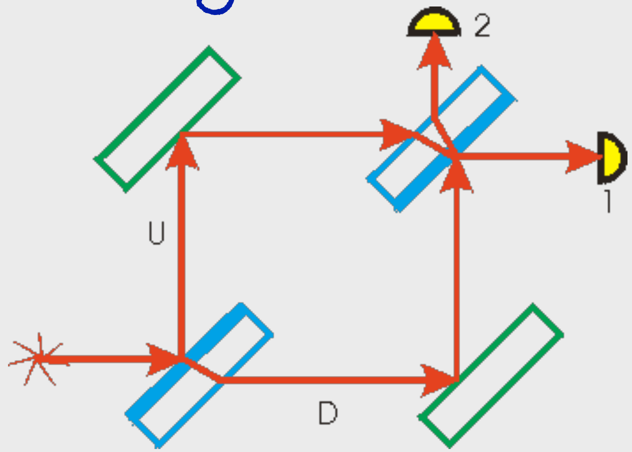

PRL **98,** 160401 (2007) PHYSICAL REVIEW LETTERS week ending 20 APRIL 2007

Optimal Quantum Estimation of Loss in Bosonic Channels

Alex Monras[1] and Matteo G. A. Paris[2,*]
[1]*Grup de Fisica Teòrica & IFAE, Universitat Autònoma de Barcelona, Bellaterra E-08193, Spain*
[2]*Dipartimento di Fisica dell'Università di Milano, Milano I-20133, Italia*
(Received 7 February 2007; published 17 April 2007)

Quantum Interferometry





use of quantum states of light to improve sensitivity

Optimization over input states (Caves 1981, Shapiro 1990)

Effects of detection noise (Paris 1995)

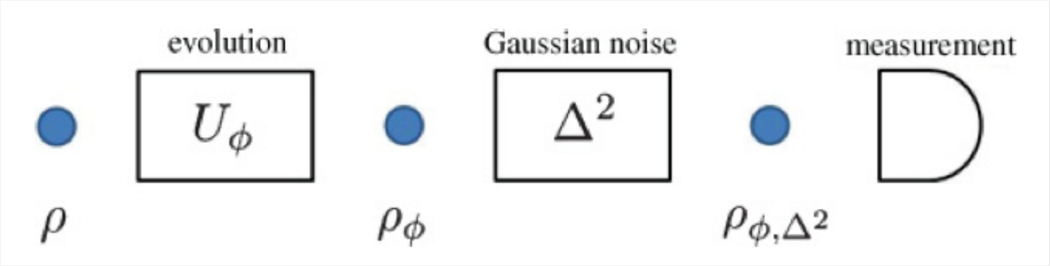
Effects of losses (Kapcprovicz et al 2010)

Multiple interference (Demkowicz-Dobrzanski et al 2009)

Noise as a resource (Braun et al 2011)

Fixed number of particles, atomic interf (Pezze et al 2005)

Estimation of phase in the presence of phase diffusion



$$U_\phi = \exp\{-i\, a^\dagger a\, \phi\}$$

$$\varrho_\phi = \mathcal{N}_\Delta[U_\phi\, \varrho\, U_\phi^\dagger] = U_\phi\, \mathcal{N}_\Delta[\varrho]U_\phi^\dagger$$

$$\mathcal{N}_\Delta[\,\varrho\,] = \sum_{nm} e^{-\Delta^2(n-m)^2}\, \varrho_{nm}\, |n\rangle\langle m|$$

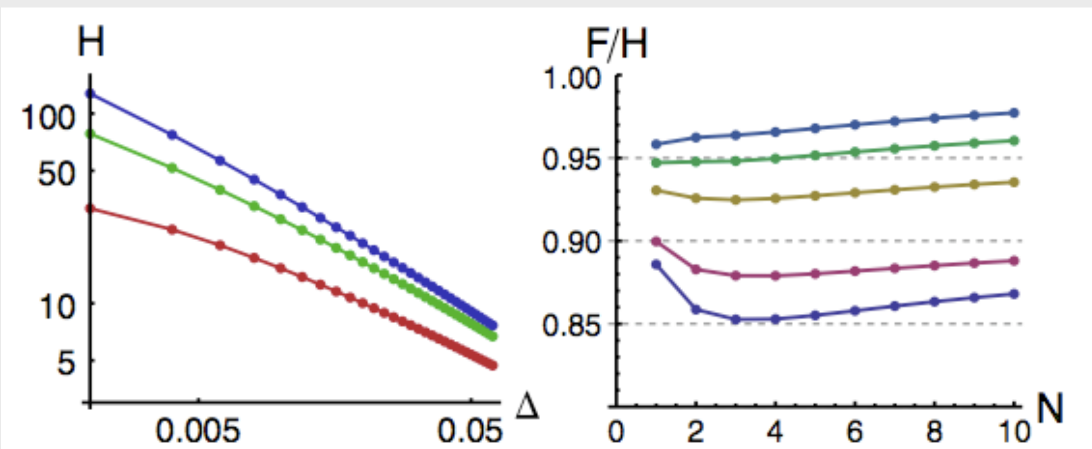
- In the noiseless case the optimal probe is the squeezed vacuum and $H = 8(N^2+N)$ (Monras 2006)

Estimation of phase in the presence of phase diffusion

- In the presence of noise we have (approximate) scaling laws

$$H(N,\Delta) \simeq k^2 H(N/k, k\Delta) \qquad \beta_{\rm opt}(N,\Delta) \simeq \beta_{\rm opt}(N/k, k\Delta)$$

- Homodyning is nearly optimal for low and high noise



PRL 106, 153603 (2011) PHYSICAL REVIEW LETTERS week ending 15 APRIL 2011

Optical Phase Estimation in the Presence of Phase Diffusion

Marco G. Genoni,[1,2] Stefano Olivares,[3] and Matteo G. A. Paris[2,*]
[1]*QOLS, Blackett Laboratory, Imperial College London, London SW7 2BW, United Kingdom*
[2]*Dipartimento di Fisica, Università degli Studi di Milano, I-20133 Milano, Italy*
[3]*Dipartimento di Fisica, Università degli Studi di Trieste, I-34151 Trieste, Italy*
(Received 12 December 2010; published 14 April 2011)

Estimation of entanglement

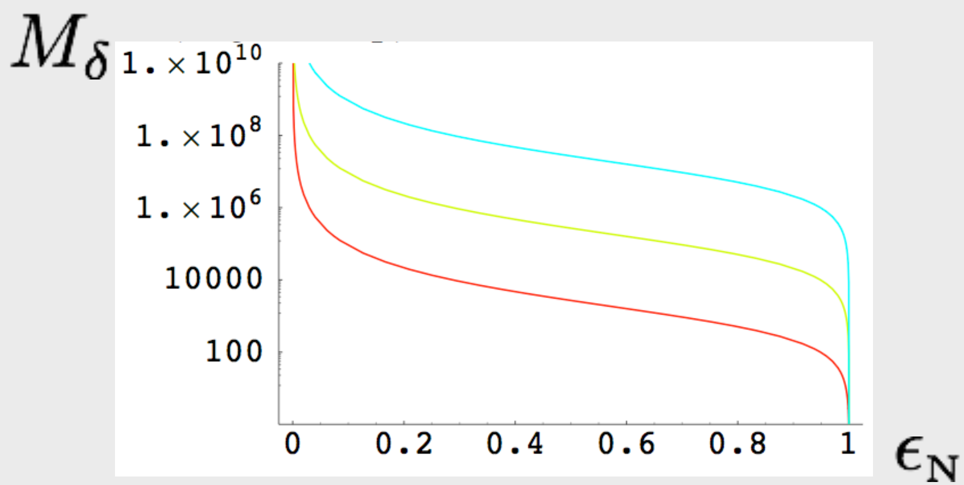
- pure states (Schmidt decomposition)

$$|\Psi_q\rangle = \sqrt{q}|0\rangle_A|0\rangle_B + \sqrt{1-q}|1\rangle_A|1\rangle_B$$

- entanglement measure: function of q (negativity, linear and VN entropy)

- QSNR is vanishing for vanishing entanglement

$$Q(\epsilon_{\rm N}) = \frac{\epsilon_{\rm N}^2}{1-\epsilon_{\rm N}^2} \overset{\epsilon_{\rm N}\to 0}{\sim} \epsilon_{\rm N}^2$$



Estimation of entanglement

- different measures (negativity, entropy, distance) and families of states (qubit and CV)

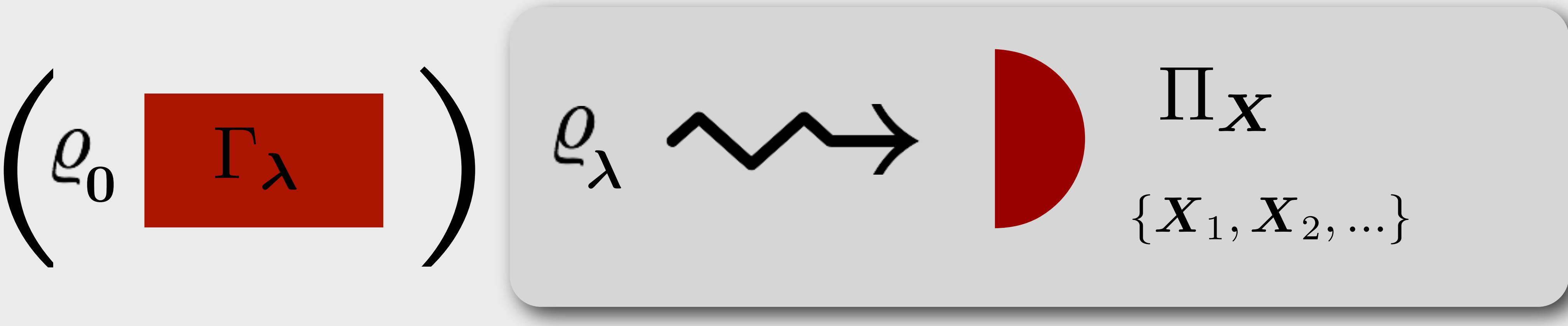
- QFI is increasing with entanglement
  QSNR diverges for maximal entanglement

- Qubit: QSNR is vanishing for vanishing entanglement
  Estimation of (low) entanglement is inherently inefficient

- CV: appropriate entanglement measure may achieve efficient estimation

M. G. Genoni, P. Giorda and M. G. A. Paris PRA 2008

The multiparametric case



$$\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots\} \qquad\qquad \boldsymbol{X} = \{x_1, x_2, \ldots\}$$

The multiparametric case

- QFI matrix $\boldsymbol{H}(\boldsymbol{\lambda})_{\mu\nu} = \mathrm{Tr}\left[\varrho_{\boldsymbol{\lambda}} \frac{L_\mu L_\nu + L_\nu L_\mu}{2}\right]$

- bound on covariance (not achievable) $\mathrm{Cov}[\gamma]_{ij} = \langle \lambda_i \lambda_j \rangle - \langle \lambda_i \rangle \langle \lambda_j \rangle$

$$\mathrm{Cov}[\gamma] \geq \frac{1}{M} \boldsymbol{H}(\boldsymbol{\lambda})^{-1}$$

- single parameter (achievable) $\mathrm{Var}(\lambda_\mu) \geq \frac{1}{M} (\boldsymbol{H}^{-1})_{\mu\mu}$

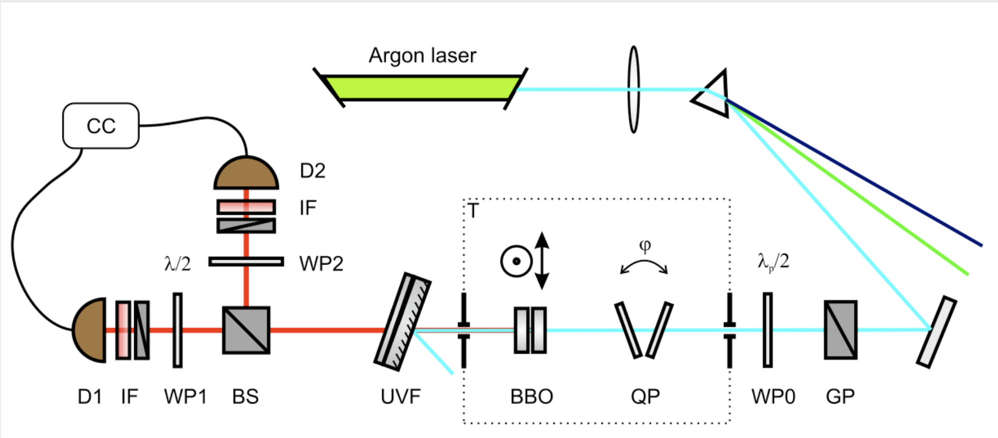
repametrization $\widetilde{\boldsymbol{\lambda}} = \{\widetilde{\lambda}_j = \widetilde{\lambda}_j(\boldsymbol{\lambda})\}$ $\widetilde{\lambda}_1 \equiv g(\boldsymbol{\lambda})$

$$\widetilde{L}_\mu = \sum_\nu B_{\mu\nu} L_\nu \qquad \widetilde{\boldsymbol{H}} = \boldsymbol{B}\boldsymbol{H}\boldsymbol{B}^T \qquad B_{\mu\nu} = \partial \lambda_\nu / \partial \widetilde{\lambda}_\mu$$

Estimation of entanglement (@INRIM)

$$|\psi_\phi\rangle = \cos\phi|HH\rangle + \sin\phi|VV\rangle$$

$$D_\phi = \cos^2\phi|HH\rangle\langle HH| + \sin^2\phi|VV\rangle\langle VV|$$



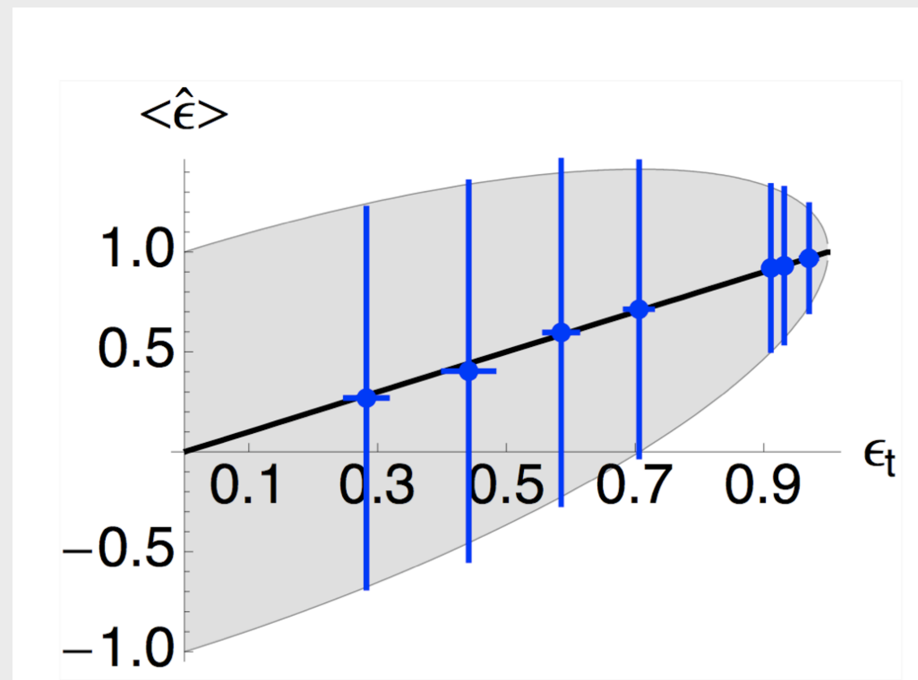
$$\varrho_\epsilon = p|\psi_\phi\rangle\langle\psi_\phi| + (1-p)D_\phi$$

$$\epsilon = p\sin 2\phi$$

optimal estimation by visibility measurements

Fisher information is monotone with entanglement

Estimation of entanglement is inherently inefficient

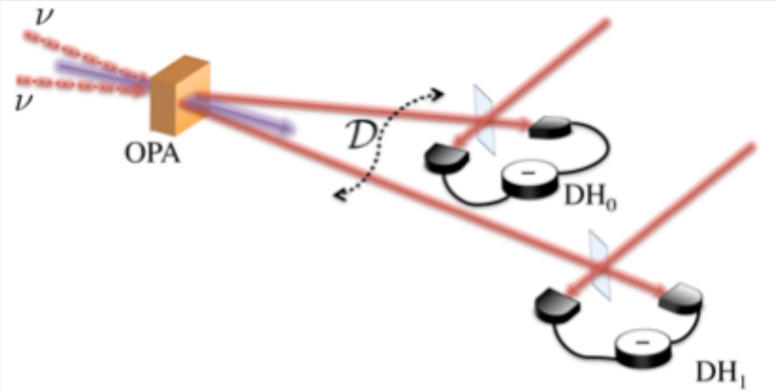


PRL 104, 100501 (2010) PHYSICAL REVIEW LETTERS week ending 12 MARCH 2010

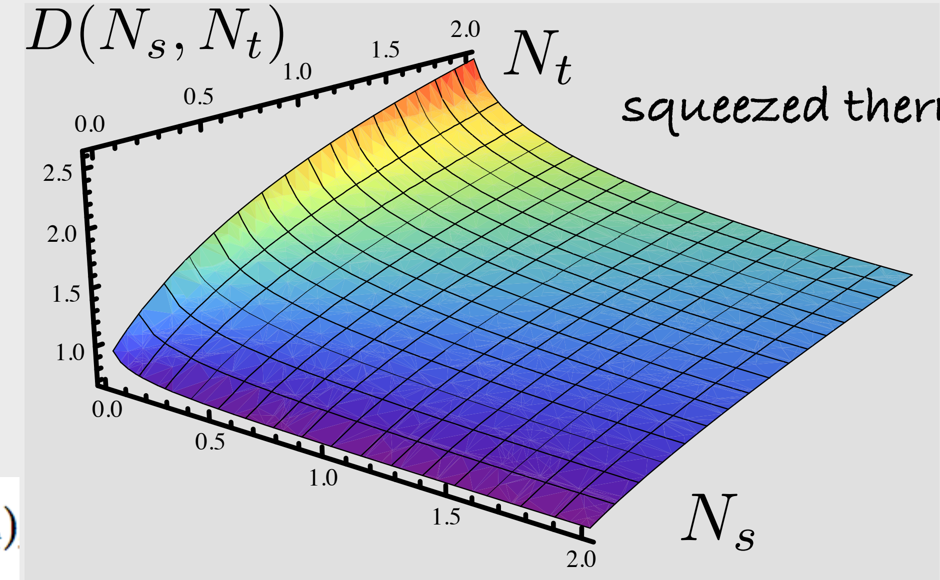
Experimental Estimation of Entanglement at the Quantum Limit

Giorgio Brida,[1] Ivo Pietro Degiovanni,[1] Angela Florio,[1,2] Marco Genovese,[1] Paolo Giorda,[3] Alice Meda,[1] Matteo G. A. Paris,[4,5] and Alexander Shurupov[6,1,7]

Estimation of quantum discord (@Charles Fabry)



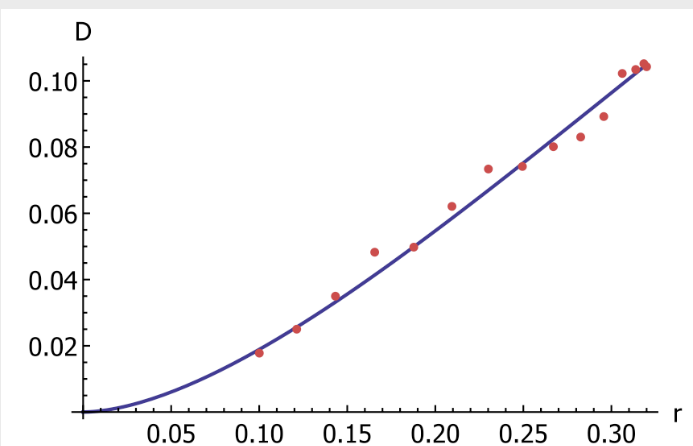
$$\varrho(N_s, N_t) = S_2(s)\nu(N_t) \otimes \nu(N_t)S_2(s)^\dagger$$

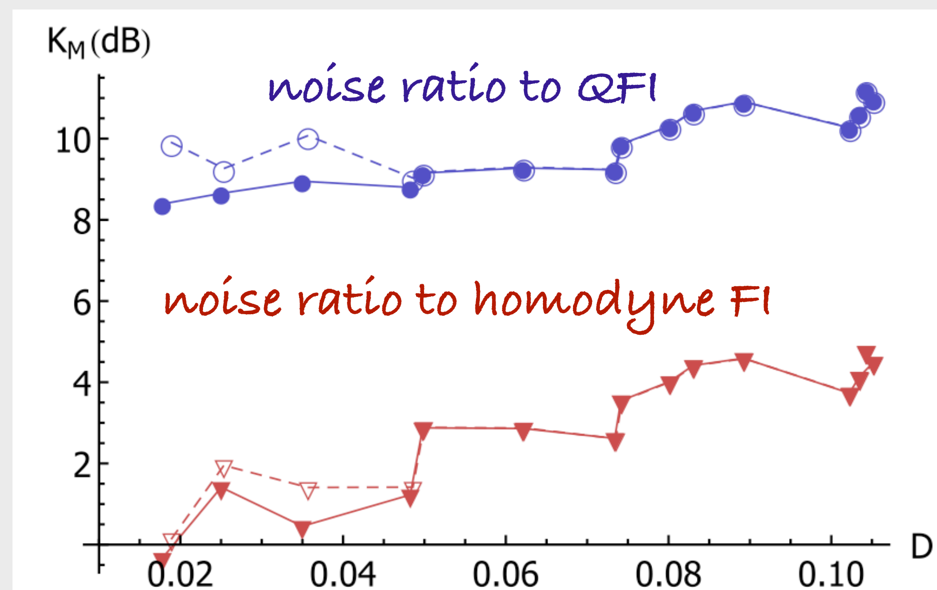


squeezed thermal states

Homodyne detection + inversion
or Bayesian analysis

$$p(N_s, N_t|X) = \frac{1}{\mathcal{N}} p(X|N_s, N_t)p_0(N_s)p_0(N_t)$$



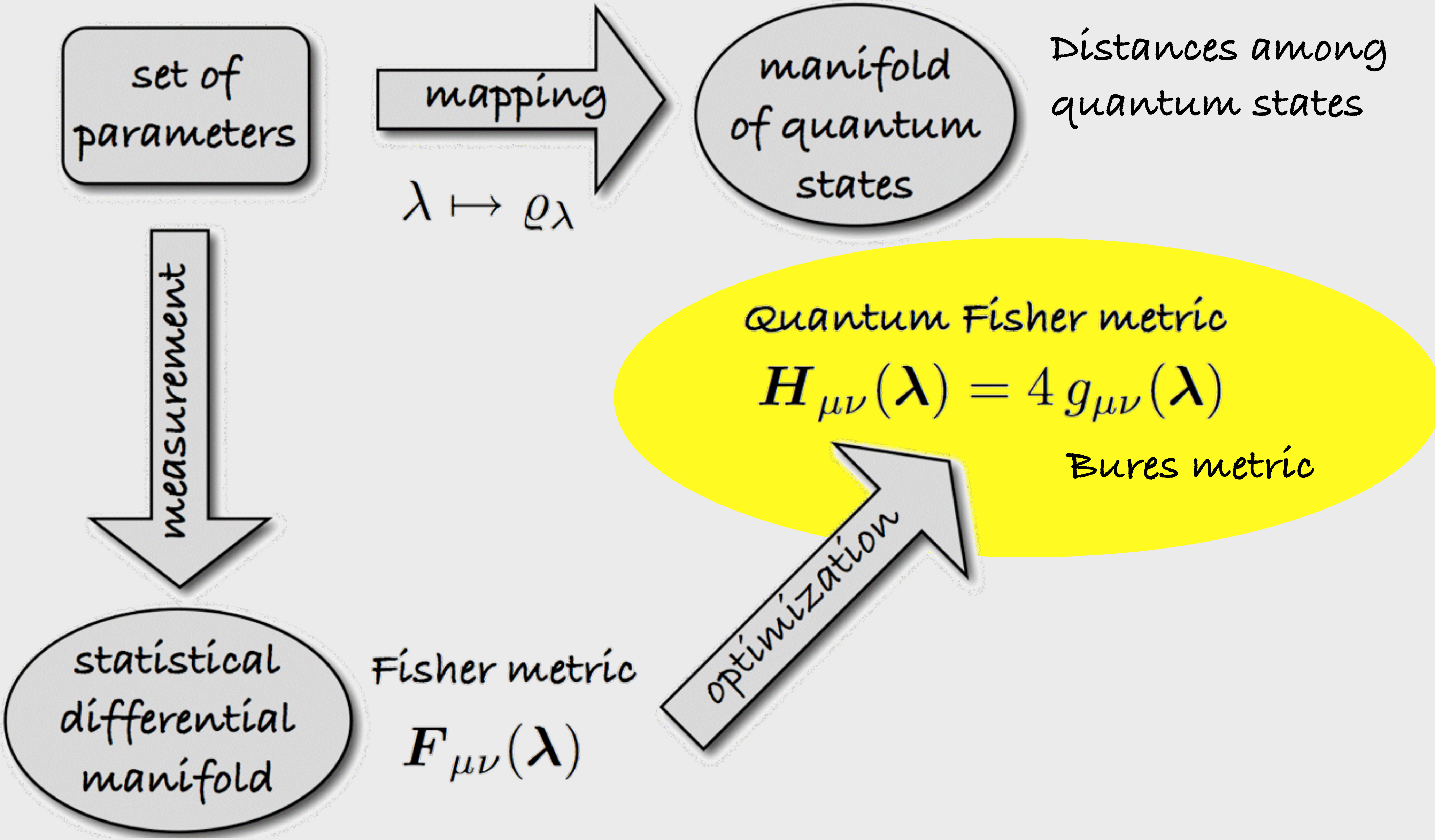


PRL 109, 180402 (2012) PHYSICAL REVIEW LETTERS week ending 2 NOVEMBER 2012

Homodyne Estimation of Gaussian Quantum Discord

Rémi Blandino,[1,*] Marco G. Genoni,[2,†] Jean Etesse,[1] Marco Barbieri,[1] Matteo G. A. Paris,[3,4]
Philippe Grangier,[1] and Rosa Tualle-Brouri[5,1]

Geometry of quantum estimation



M. G. A. Paris IJQI 2008

Bures distance

- Bures distance in terms of fidelity

$$D_B^2(\varrho,\sigma) = 2[1-\sqrt{F(\varrho,\sigma)}] \qquad F(\varrho,\sigma) = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\varrho}\sigma\sqrt{\varrho}}\right]\right)^2$$

$$d_B^2 = D_B^2(\varrho_{\boldsymbol{\lambda}}, \varrho_{\boldsymbol{\lambda}+d\boldsymbol{\lambda}}) = g_{\mu\nu} d\lambda_\mu d\lambda_\nu$$

reduces to overlap
for pure states

Quantum phase transitions

- Phase transitions: strong change in some relevant observable $\rightsquigarrow$ change in the state of the system

- Quantum phase transitions: strong departure from the initial previous density matrix $\rightsquigarrow$ increase of distance between infinitesimally close states
  (Zanardi et al. PRL 2007, PRA 2007)

QPT $\rightsquigarrow$ extremal points of Bures metric

Criticality and estimation

- Estimation is very effective at critical points (useful in systems with a tuning parameter, eg interacting spins in an external field)

- quantum: coupling constants

  $G_\lambda \sim L^\alpha$ at zero temperature

  $G_\lambda \sim T^{-\beta}$

  classical: temperature

  $G_\beta \sim \beta^{-2} C_V(\beta)$

P. Zanardi, M. G. A. Paris, L. Campos Venuti PRA 2008

L. Campos Venuti, C. Invernizzi, M. Korbman and M. G. A. Paris PRA 2008

(REM QFI=G instead of H to avoid confusion with the Hamiltonian)

Criticality and estimation: Ising model

$$H = -J \sum_{k=1}^{L} \sigma_k^x \sigma_{k+1}^x - h \sum_{k=1}^{L} \sigma_k^z,$$

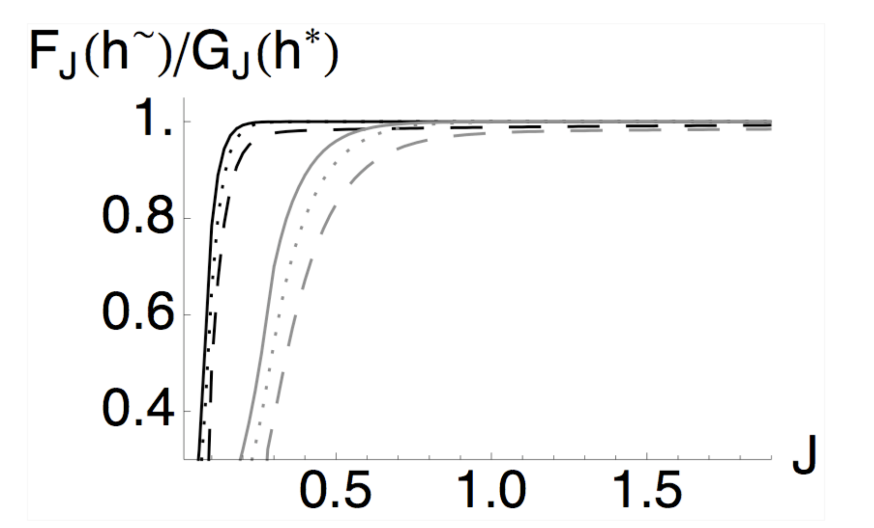
- QFI at zero temperature is maximized by J=h for any L (at regular points we have only extensive behaviour)

$$G_J(L, T=0, h^* = J) \simeq \frac{L^2}{8J^2} + O(L).$$

QSNR independent on the value of J

- Ultimate bounds to precision may be achieved by measurement of total magnetization

$$M_z = \frac{1}{L} \sum_i \sigma_i^z$$

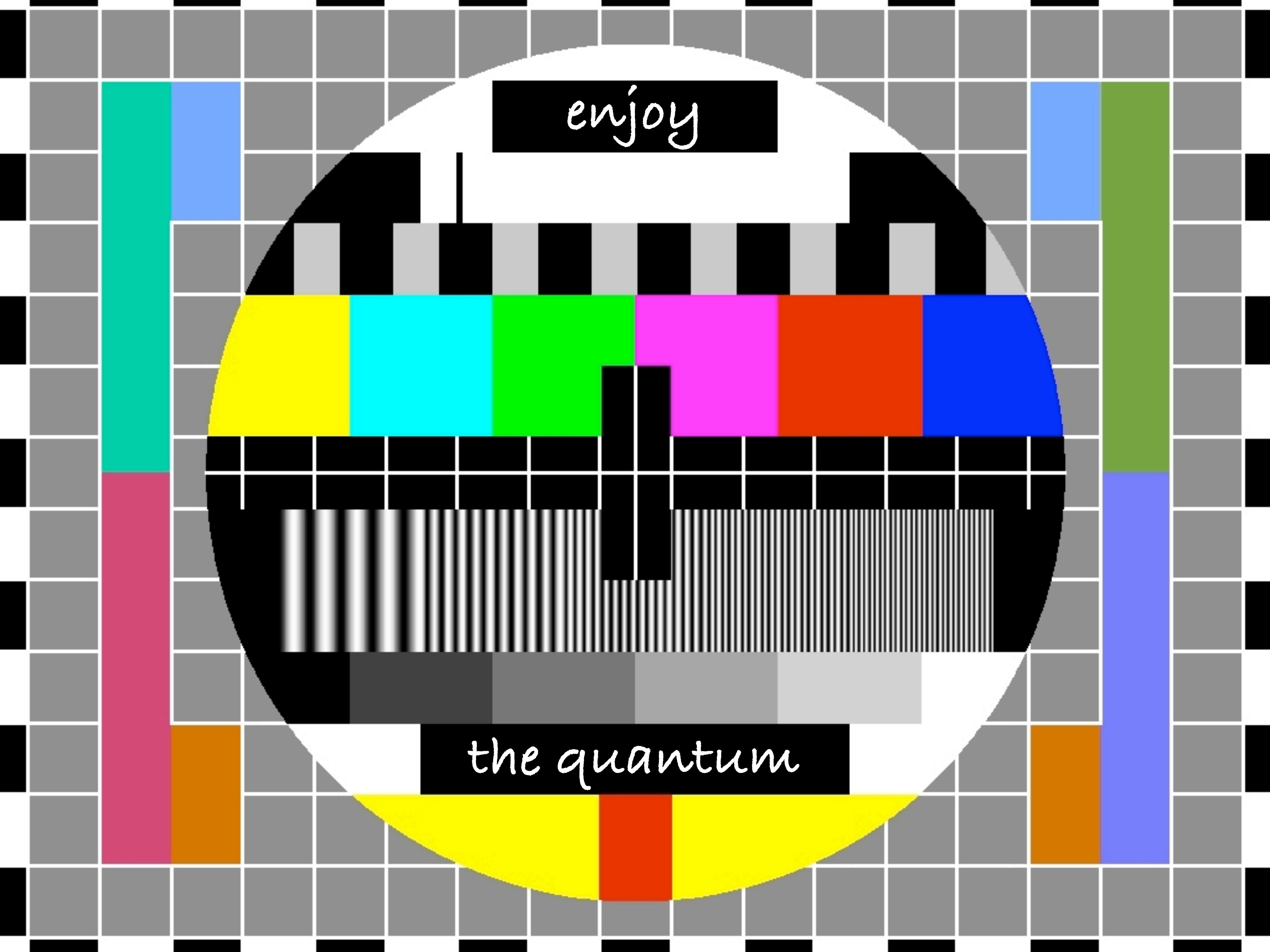


Summary

- Quantum estimation: foundations & quantum technology
  - Optimal quantum estimator in terms of SLD
  - Ultimate bounds to the precision of the estimation of any quantity of interest including non-observables: entanglement, discord...
  - intrinsic estimability of a parameter
  - classical and quantum contributions to uncertainty

- Quantum Fisher information is proportional to Bures metric:
  - Criticality is a resource
  - Superextensive behaviour with size and large estimability

Matteo G. A. Paris
Quantum estimation for quantum technology
Int. J. Quant. Inf. 7, 125 (2009) arXiv:0804.2981


enjoy
the quantum